SOBOLEV-LORENTZ SPACES IN THE EUCLIDEAN SETTING AND COUNTEREXAMPLES

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Abstract. This paper studies the inclusions between different Sobolev-Lorentz spaces $W^{1,(p,q)}(\Omega)$ defined on open sets $\Omega \subset \mathbf{R}^n$, where $n \geq 1$ is an integer, $1 < p < \infty$ and $1 \leq q \leq \infty$. We prove that if $1 \leq q < r \leq \infty$, then $W^{1,(p,q)}(\Omega)$ is strictly included in $W^{1,(p,r)}(\Omega)$.

We show that although $H^{1,(p,\infty)}(\Omega) \subsetneq W^{1,(p,\infty)}(\Omega)$ where $\Omega \subset \mathbf{R}^n$ is open and $n \geq 1$, there exists a partial converse. Namely, we show that if a function u in $W^{1,(p,\infty)}(\Omega), n \geq 1$ is such that u and its distributional gradient ∇u have absolutely continuous (p,∞)-norm, then u belongs to $H^{1,(p,\infty)}(\Omega)$ as well.

We also extend the Morrey embedding theorem to the Sobolev-Lorentz spaces $H_0^{1,(p,q)}(\Omega)$ with $1 \leq n < p < \infty$ and $1 \leq q \leq \infty$. Namely, we prove that the Sobolev-Lorentz spaces $H_0^{1,(p,q)}(\Omega)$ embed into the space of Hölder continuous functions on $\overline{\Omega}$ with exponent $1 - \frac{n}{p}$ whenever $\Omega \subset \mathbf{R}^n$ is open, $1 \leq n < p < \infty$, and $1 \leq q \leq \infty$.

1. Introduction

In this paper we study the Sobolev-Lorentz spaces in the Euclidean setting and the inclusions between them. This paper is motivated by the results obtained in my 2006 PhD thesis [6] and in my book [9]. There I studied the Sobolev-Lorentz spaces and the associated Sobolev-Lorentz capacities in the Euclidean setting for $n \geq 2$. The restriction on n there was due to the the fact that I studied the n, q-capacity for $n > 1$.

The Sobolev-Lorentz spaces have also been studied by Cianchi-Pick in [4] and [5], by Kauhanen-Koskela-Malý in [22], and by Malý-Swanson-Ziemer in [25].

The classical Sobolev spaces were studied by Gilbarg-Trudinger in [15], Maz'ja in [26], Evans in [12], Heinonen-Kilpeläinen-Martio in [19], and by Ziemer in [30].

The Lorentz spaces were studied by Bennett-Sharpley in [1], Hunt in [21], and by Stein-Weiss in [29].

The Newtonian Sobolev spaces in the metric setting were studied by Shanmugalingam in [27] and [28]. See also Heinonen [18]. Costea-Miranda studied the Newtonian Lorentz Sobolev spaces and the corresponding global p, q-capacities in [11].

There are several other definitions of Sobolev-type spaces in the metric setting when $p = q$; see Hajłasz [16], [17], Heinonen-Koskela [20], Cheeger [3], and Franchi-Hajłasz-Koskela [14]. It has been shown that under reasonable hypotheses, the majority of these definitions yields the same space; see Franchi-Hajłasz-Koskela [14] and Shanmugalingam [27].

The Sobolev-Lorentz relative p, q-capacity was studied in the Euclidean setting by Costea (see [6], [7] and [9]) and by Costea-Maz'ya [10]. The Sobolev p-capacity was studied by Maz'ya [26] and by Heinonen-Kilpeläinen-Martio [19] in $\mathbf{R}^n$ and by J. Björn [2], Costea [8] and Kinnunen-Martio [23] and [24] in metric spaces.

The Sobolev-Lorentz spaces can be also studied in the Euclidean setting for $n = 1$. We do it in this paper. Many of the results on Sobolev-Lorentz spaces that we obtained in [6] and [9] in dimension $n \geq 2$ were extended here to the case $n = 1$.

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In Section 3 we start by presenting some of the basic properties of the Lorentz spaces $L^{p,q}(\Omega;\mathbf{R}^m)$, where $\Omega\subset\mathbf{R}^n$ is open, $n,m\geq 1$ are integers, $1<p<\infty$ and $1\leq q\leq\infty$.

It is known that $L^{p,q}((0,\Omega_n r^n))\subsetneq L^{p,s}((0,\Omega_n r^n))$. We see this in Theorem 3.4 by constructing a function u in $L^{p,s}((0,\Omega_n r^n))\setminus L^{p,q}((0,\Omega_n r^n))$. Here $r>0$, $n\geq 1$, $1<p<\infty$ and $1\leq q<s\leq\infty$.

This function u is used in Theorem 3.5 to construct a radial function v that is smooth in the punctured ball $B^*(0,r)$ such that $|\nabla v|$ is in $L^{p,s}(B(0,r))\setminus L^{p,q}(B(0,r))$. Later it will be shown in Theorem 4.13 that v is in $W^{1,(p,s)}(B(0,r))\setminus W^{1,(p,q)}(B(0,r))$. This shows that the inclusion $W^{1,(p,q)}(B(0,r))\subset W^{1,(p,s)}(B(0,r))$ is strict whenever $r>0$, $n\geq 1$, $1<p<\infty$ and $1\leq q<s\leq\infty$.

In Section 4 we revisit many of the results from my PhD thesis [6, Chapter V] and from my book [9, Chapter 3] and we extend them to the case $n=1$. We improve some of the old results from [6, Chapter V] and from [9, Chapter 3].

We also obtain some new results in this section. Among them we mention the case $q=\infty$ for Theorems 4.11 and 4.12 (see the discussion below) as well as the strict inclusion $W^{1,(p,q)}(B(0,r))\subsetneq W^{1,(p,s)}(B(0,r))$ that we discussed above. As before, $r>0$, $n\geq 1$, $1<p<\infty$ and $1\leq q<s\leq\infty$ (see Theorem 4.13).

For $n\geq 2$, we proved in Costea [6] and [9] (by using partition of unity and convolution) that $H^{1,(p,q)}(\Omega)=W^{1,(p,q)}(\Omega)$ whenever $1<p<\infty$ and $1\leq q<\infty$. The partition of unity and convolution technique used there is similar to the techniques used by Ziemer in [30] and by Heinonen-Kilpeläinen-Martio in [19].

We proved in [6] and [9] (for $n\geq 2$) that $H^{1,(p,\infty)}(\Omega)\subsetneq W^{1,(p,\infty)}(\Omega)$. Once we constructed a function $u\in W^{1,(p,\infty)}(\Omega)$ such that its distributional gradient ∇u did not have an absolutely continuous (p,∞)-norm, we proved there that u was not in $H^{1,(p,\infty)}(\Omega)$.

In Section 4 of this paper, Proposition 4.7 and Theorem 4.8 show that $H^{1,(p,\infty)}(\Omega)\subsetneq W^{1,(p,\infty)}(\Omega)$ for $n\geq 1$. In this paper we also give a partial converse. Namely, we show in Theorem 4.11 that if a function u in $W^{1,(p,q)}(\Omega), n\geq 1$, $1\leq q\leq\infty$ is such that u and its distributional gradient ∇u have absolutely continuous (p,q)-norm, then u belongs to $H^{1,(p,q)}(\Omega)$ as well. This result is new for $q=\infty$ and $n\geq 1$ and improves a result from [6] and [9], proved there for $n\geq 2$ and $1\leq q<\infty$. We proved this result via a partition of unity and convolution argument, because convolution and partition of unity work well on functions u that have absolutely continuous (p,q)-norm along with their distributional gradients ∇u.

In Theorem 4.12 we show that if a function u in $W^{1,(p,q)}(\mathbf{R}^n), n\geq 1$ is such that u and its distributional gradient ∇u have absolutely continuous (p,q)-norm, $1\leq q\leq\infty$, then u belongs to $H_0^{1,(p,q)}(\mathbf{R}^n)$ as well. This result is new when $q=\infty$ and $n\geq 1$ and improves a result from [6] and [9], proved there for $n\geq 2$ and $1\leq q<\infty$.

In Section 5 (among other things) we prove the Morrey embedding theorem for the Sobolev-Lorentz spaces $H_0^{1,(p,q)}(\Omega)$.

For $n=1$, we prove in Theorem 5.5 that if $\Omega\subset\mathbf{R}$ is an open interval, then $H_0^{1,(p,q)}(\Omega)$ and $W^{1,(p,q)}(\Omega)$ embed into the space of Hölder continuous functions in $\overline{\Omega}$ with exponent $1-\frac{1}{p}$.

For $1<n<p<\infty$, we prove in Theorem 5.6 (among other things) that the spaces $H_0^{1,(p,q)}(\Omega)$ and $C_0(\Omega)\cap W^{1,(p,q)}(\Omega)$ embed into the space $C^{0,1-\frac{n}{p}}(\overline{\Omega})$ of Hölder continuous functions in $\overline{\Omega}$ with exponent $1-\frac{n}{p}$ whenever $\Omega\subset\mathbf{R}^n$ is open and $1\leq q\leq\infty$.

Since we deal with functions in $H_0^{1,(p,q)}(\Omega)$ or in $C_0(\Omega)\cap W^{1,(p,q)}(\Omega)$ when $1<n<p<\infty$ and $1\leq q\leq\infty$, no regularity assumptions on $\partial\Omega$ are needed.

When $1<n<p<\infty$, we first prove the Morrey embedding $C_0(\Omega)\cap W^{1,(p,q)}(\Omega)\hookrightarrow C^{0,1-\frac{n}{p}}(\overline{\Omega})$. The embedding $H_0^{1,(p,q)}(\Omega)\hookrightarrow C^{0,1-\frac{n}{p}}(\overline{\Omega})$ follows afterwards, after an approximation argument with functions from $C_0^\infty(\Omega)$. We also rely on the well-known Poincaré inequality in the Euclidean setting and we invoke the classical embedding for $1<n<s<p<\infty$, proved by Evans in [12] and by Gilbarg-Trudinger in [15].

2. Notations

We recall the standard notation to be used throughout this paper. Throughout this paper, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line. $C(a, b, \ldots)$ is a constant that depends only on the parameters $a, b, \cdots$.

Throughout this paper Ω will denote a nonempty open subset of $\mathbf{R}^n$, while $dx = dm_n(x)$ will denote the Lebesgue n-measure in $\mathbf{R}^n$, where $n \geq 1$ is integer. For $E \subset \mathbf{R}^n$, the boundary, the closure, and the complement of E with respect to $\mathbf{R}^n$ will be denoted by ∂E, $\overline{E}$, and $\mathbf{R}^n \setminus E$, respectively, while $|E| = \int_E dx$ will denote the measure of E whenever E is measurable; diam E is the Euclidean diameter of E and $E \subset\subset F$ means that $\overline{E}$ is a compact subset of F.

Moreover, $B(a, r) = \{x \in \mathbf{R}^n : |x - a| < r\}$ is the open ball with center $a \in \mathbf{R}^n$ and radius $r > 0$, $B^*(a, r) = \{x \in \mathbf{R}^n : 0 < |x - a| < r\}$ is the punctured open ball with center $a \in \mathbf{R}^n$ and radius $r > 0$, while $\overline{B}(a, r) = \{x \in \mathbf{R}^n : |x - a| \leq r\}$ is the closed ball with center $a \in \mathbf{R}^n$ and radius $r > 0$.

For two sets $A, B \subset \mathbf{R}^n$, we define $\operatorname{dist}(A, B)$, the distance between A and B, by

$$\operatorname{dist}(A, B) = \inf_{a \in A, b \in B} |a - b|.$$

For $n \geq 1$ integer, Ω_n denotes the Lebesgue measure of the n-dimensional unit ball. (That is, $\Omega_n = |B(0, 1)|$). For $n \geq 2$ integer, ω_{n-1} denotes the spherical measure of the $n-1$-dimensional sphere; thus, $\omega_{n-1} = n\Omega_n$ for every integer $n \geq 2$.

For $\Omega \subset \mathbf{R}^n$, $C(\Omega)$ is the set of all continuous functions $u : \Omega \to \mathbf{R}$, while $C(\overline{\Omega})$ is the set of all continuous functions $u : \overline{\Omega} \to \mathbf{R}$. Moreover, for a measurable $u : \Omega \to \mathbf{R}$, supp u is the smallest closed set such that u vanishes outside supp u. We also define

$$\begin{aligned}
C^k(\Omega) &= \{\varphi : \Omega \to \mathbf{R} : \text{ the } k\text{th-derivative of } \varphi \text{ is continuous}\} \\
C_0^k(\Omega) &= \{\varphi \in C^k(\Omega) : \operatorname{supp} \varphi \subset\subset \Omega\} \\
C^\infty(\Omega) &= \bigcap_{k=1}^{\infty} C^k(\Omega) \\
C_0^\infty(\Omega) &= \{\varphi \in C^\infty(\Omega) : \operatorname{supp} \varphi \subset\subset \Omega\}.
\end{aligned}$$

For a function $\varphi \in C^\infty(\Omega)$ we write

$$\nabla \varphi = (\partial_1 \varphi, \partial_2 \varphi, \ldots, \partial_n \varphi)$$

for the gradient of φ.

Let $f : \Omega \to \mathbf{R}$ be integrable. For $E \subset \Omega$ measurable with $0 < |E| < \infty$, we define

$$f_E = \frac{1}{|E|} \int_E f dx.$$

For a measurable vector-valued function $f = (f_1, \ldots, f_m) : \Omega \to \mathbf{R}^m$, we let

$$|f| = \sqrt{f_1^2 + f_2^2 + \ldots + f_m^2}.$$

$L^\infty(\Omega; \mathbf{R}^m)$ denotes the space of essentially bounded measurable functions $u : \Omega \to \mathbf{R}^m$ with

$$||u||_{L^\infty(\Omega)} = \operatorname{ess\,sup} |u| < \infty.$$

3. Lorentz Spaces

3.1. **Definitions and basic properties.** Let $f : \Omega \to \mathbf{R}$ be a measurable function. We define $\lambda_{[f]}$, the *distribution function* of f as follows (see Bennett-Sharpley [1, Definition II.1.1] and Stein-Weiss [29, p. 57]):

$$\lambda_{[f]}(t) = |\{x \in \Omega : |f(x)| > t\}|, \qquad t \geq 0.$$

We define f^*, the *nonincreasing rearrangement* of f by

$$f^*(t) = \inf\{v : \lambda_{[f]}(v) \leq t\}, \quad t \geq 0.$$

(See Bennett-Sharpley [1, Definition II.1.5] and Stein-Weiss [29, p. 189]). We notice that f and f^* have the same distribution function. Moreover, for every positive α we have $(|f|^\alpha)^* = (|f|^*)^\alpha$ and if $|g| \leq |f|$ a.e. on Ω, then $g^* \leq f^*$. (See Bennett-Sharpley [1, Proposition II.1.7]). We also define f^{**}, the *maximal function* of f^* by

$$f^{**}(t) = m_{f^*}(t) = \frac{1}{t}\int_0^t f^*(s)ds, \quad t > 0.$$

(See Bennett-Sharpley [1, Definition II.3.1] and Stein-Weiss [29, p. 203]).

Throughout this paper, we denote by p' the Hölder conjugate of $p \in [1, \infty]$, that is

$$p' = \begin{cases} \infty & \text{if } p = 1 \\ \frac{p}{p-1} & \text{if } 1 < p < \infty \\ 1 & \text{if } p = \infty. \end{cases}$$

The *Lorentz space* $L^{p,q}(\Omega)$, $1 < p < \infty$, $1 \leq q \leq \infty$, is defined as follows:

$$L^{p,q}(\Omega) = \{f : \Omega \to \mathbf{R} : f \text{ is measurable and } ||f||_{L^{p,q}(\Omega)} < \infty\},$$

where

$$||f||_{L^{p,q}(\Omega)} = ||f||_{p,q} = \begin{cases} \left(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \, \frac{dt}{t}\right)^{\frac{1}{q}} & 1 \leq q < \infty \\ \sup_{t>0} t\lambda_{[f]}(t)^{\frac{1}{p}} = \sup_{s>0} s^{\frac{1}{p}} f^*(s) & q = \infty. \end{cases}$$

(See Bennett-Sharpley [1, Definition IV.4.1] and Stein-Weiss [29, p. 191]). If $1 \leq q \leq p$, then $||\cdot||_{L^{p,q}(\Omega)}$ already represents a norm, but for $p < q \leq \infty$ it represents a quasinorm that is equivalent to the norm $||\cdot||_{L^{(p,q)}(\Omega)}$, where

$$||f||_{L^{(p,q)}(\Omega)} = ||f||_{(p,q)} = \begin{cases} \left(\int_0^\infty (t^{\frac{1}{p}} f^{**}(t))^q \, \frac{dt}{t}\right)^{\frac{1}{q}} & 1 \leq q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t) & q = \infty. \end{cases}$$

(See Bennett-Sharpley [1, Definition IV.4.4]).

Namely, from Lemma IV.4.5 in Bennett-Sharpley [1] we have that

$$||f||_{L^{p,q}(\Omega)} \leq ||f||_{L^{(p,q)}(\Omega)} \leq \frac{p}{p-1}||f||_{L^{p,q}(\Omega)}$$

for every $1 \leq q \leq \infty$.

For a measurable vector-valued function $f = (f_1, \ldots, f_m) : \Omega \to \mathbf{R}^m$ we say that $f \in L^{p,q}(\Omega; \mathbf{R}^m)$ if and only if $f_i \in L^{p,q}(\Omega)$ for $i = 1, 2, \ldots, m$, if and only if $|f| \in L^{p,q}(\Omega)$ and we define

$$||f||_{L^{p,q}(\Omega;\mathbf{R}^m)} = ||\,|f|\,||_{L^{p,q}(\Omega)}.$$

Similarly

$$||f||_{L^{(p,q)}(\Omega;\mathbf{R}^m)} = ||\,|f|\,||_{L^{(p,q)}(\Omega)}.$$

Obviously, it follows from the real-valued case that

$$||f||_{L^{p,q}(\Omega;\mathbf{R}^m)} \leq ||f||_{L^{(p,q)}(\Omega;\mathbf{R}^m)} \leq \frac{p}{p-1}||f||_{L^{p,q}(\Omega;\mathbf{R}^m)}$$

for every $1 \leq q \leq \infty$, and like in the real-valued case, $||\cdot||_{L^{p,q}(\Omega;\mathbf{R}^m)}$ is already a norm when $1 \leq q \leq p$, while it is a quasinorm when $p < q \leq \infty$.

It is known that $(L^{p,q}(\Omega; \mathbf{R}^m), ||\cdot||_{L^{p,q}(\Omega;\mathbf{R}^m)})$ is a Banach space for $1 \leq q \leq p$, while $(L^{p,q}(\Omega; \mathbf{R}^m), ||\cdot||_{L^{(p,q)}(\Omega;\mathbf{R}^m)})$ is a Banach space for $1 < p < \infty$, $1 \leq q \leq \infty$. These spaces are reflexive if $1 < q < \infty$ and the dual of $L^{p,q}\Omega; \mathbf{R}^m)$ is, up to equivalence of norms, the space $L^{p',q'}(\Omega; \mathbf{R}^m)$ for $1 \leq q < \infty$. See Bennett-Sharpley [1, Theorem IV.4.7, Corollaries I.4.3 and IV.4.8], Hunt [21, p. 259-262], the definition of $L^{p,q}(\Omega; \mathbf{R}^m)$ and the discussion after Proposition 3.2.

3.2. **Absolute continuity of the (p,q)-norm and reflexivity of the Lorentz spaces.**

Definition 3.1. (See Bennett-Sharpley [1, Definition I.3.1]). Let $n, m \geq 1$ be two integers, $1 < p < \infty$ and $1 \leq q \leq \infty$. Suppose $\Omega \subset \mathbf{R}^n$ is open. Let $X = L^{p,q}(\Omega; \mathbf{R}^m)$. A function f in X is said to have *absolutely continuous norm* in X if and only if $||f\chi_{E_k}||_X \to 0$ for every sequence E_k satisfying $E_k \to \emptyset$ a.e.

The following proposition gives a characterization of functions with absolutely continuous norm in $X = L^{p,q}(\Omega; \mathbf{R}^m)$.

Proposition 3.2. (See Bennett-Sharpley [1, Proposition I.3.6]). *A function f in X has absolutely continuous norm if and only if the following condition holds: whenever f_k $(k = 1, 2, \ldots)$, and g are measurable functions satisfying $|f_k| \leq |f|$ for all k and $f_k \to g$ a.e., then $||f_k - g||_X \to 0$.*

Let X_a be the subspace of X consisting of functions with absolutely continuous norm and let X_b be the closure in X of the set of simple functions. It is known that $X_a = X_b$ when $X = L^{p,q}(\Omega; \mathbf{R}^m)$ for $1 < p < \infty$, $1 \leq q \leq \infty$, and $m \geq 1$ integer. (See Bennett-Sharpley [1, Theorem I.3.13]). Moreover, we have $X_a = X_b = X$ when $X = L^{p,q}(\Omega; \mathbf{R}^m)$ for $1 < p < \infty$, $1 \leq q < \infty$, and $m \geq 1$ integer. (See Bennett-Sharpley [1, Theorem IV.4.7 and Corollary IV.4.8] and the definition of $L^{p,q}(\Omega; \mathbf{R}^m)$).

From Proposition 3.6 it follows that $X_a \subsetneq X$ for $X = L^{p,\infty}(\Omega; \mathbf{R}^m)$ whenever $m \geq 1$ is an integer. Since $L^{p,\infty}(\Omega; \mathbf{R}^m)$ can be identified with $(L^{p',1}(\Omega; \mathbf{R}^m))^*$ (see Bennett-Sharpley [1, Corollary IV.4.8] and the definition of $L^{p,q}(\Omega; \mathbf{R}^m)$), it follows from Bennett-Sharpley [1, Corollaries I.4.3, I.4.4, IV.4.8 and Theorem IV.4.7] that neither $L^{p,1}(\Omega; \mathbf{R}^m)$, nor $L^{p,\infty}(\Omega; \mathbf{R}^m)$ are reflexive whenever $1 < p < \infty$.

3.3. **Strict inclusions between Lorentz spaces.**

Remark 3.3. It is known (see Bennett-Sharpley [1, Proposition IV.4.2]) that for every $p \in (1, \infty)$ and $1 \leq r < s \leq \infty$ there exists a constant $C(p, r, s) > 0$ such that

$$||f||_{L^{p,s}(\Omega)} \leq C(p, r, s)||f||_{L^{p,r}(\Omega)} \tag{1}$$

for all measurable functions $f \in L^{p,r}(\Omega)$. In particular, $L^{p,r}(\Omega) \subset L^{p,s}(\Omega)$. Like in the real-valued case, it follows that

$$||f||_{L^{p,s}(\Omega;\mathbf{R}^m)} \leq C(p, r, s)||f||_{L^{p,r}(\Omega;\mathbf{R}^m)}$$

for every $m \geq 1$ integer and for all measurable functions $f \in L^{p,r}(\Omega; \mathbf{R}^m)$, where $C(p, r, s)$ is the constant from (1). In particular,

$$L^{p,r}(\Omega; \mathbf{R}^m) \subset L^{p,s}(\Omega; \mathbf{R}^m) \text{ for every } m \geq 1 \text{ integer.}$$

The above inclusion is strict. (See Ziemer [30, p. 37, Exercise 1.7]). Given an open ball $B(0, r) \subset \mathbf{R}^n$, where $n \geq 1$ integer, $r > 0$ and $1 \leq q_1 < q_2 \leq \infty$, we construct next in Theorem 3.4 a function $u \in L^{p,q_2}(B(0, r); \mathbf{R}^m) \setminus L^{p,q_1}(B(0, r); \mathbf{R}^m)$. In addition, Theorem 3.4 will allow us to construct later a radial function v that is smooth in a punctured ball $B^*(0, r)$ such that $|\nabla v|$ is in $L^{p,q_2}(B(0, r)) \setminus L^{p,q_1}(B(0, r))$. It is enough to assume that $m = 1$ when proving this strict inclusion.

Theorem 3.4. *Let $n \geq 1$ be an integer. Let $0 < \alpha \leq 1$ and $r > 0$. Suppose $1 < p < \infty$ and $1 \leq q_1 < q_2 \leq \infty$. We define $u_{r,\alpha,p}$ on $[0, \Omega_n r^n)$ by*

$$u_{r,\alpha,p}(t) = t^{-\frac{1}{p}} \ln^{-\alpha}\left(\frac{\Omega_n r^n e^{p\alpha}}{t}\right). \tag{2}$$

We also define

$$u_{rad,r,\alpha,p} : [0, r) \to [0, \infty], u_{rad,r,\alpha,p}(t) := u_{r,\alpha,p}(\Omega_n t^n) \text{ and} \tag{3}$$

$$u_{r,\alpha,n,p} : B(0, r) \subset \mathbf{R}^n \to [0, \infty], u_{r,\alpha,n,p}(x) := u_{rad,r,\alpha,p}(|x|). \tag{4}$$

Then

(i) *$u_{r,\alpha,p}$ is a decreasing function on $[0, \Omega_n r^n)$ and*

$$u^*_{r,\alpha,n,p}(t) = u^*_{r,\alpha,p}(t) = u_{r,\alpha,p}(t) \text{ for all } t \in [0, \Omega_n r^n). \tag{5}$$

(ii) $u_{r,\alpha,n,p} \in L^{p,q_2}(B(0,r)) \setminus L^{p,q_1}(B(0,r))$ *if* $1 \le q_1 \le \frac{1}{\alpha} < q_2 \le \infty$.

Proof. (i) Since $u_{r,\alpha,p}$ is defined on $[0, \Omega_n r^n)$, it follows that $u^*_{r,\alpha,p}(t) = 0$ whenever $\Omega_n r^n \le t < \infty$. Similarly, since $u_{r,\alpha,n,p}$ is defined on $B(0,r)$ and $|B(0,r)| = \Omega_n r^n$, it follows that $u^*_{r,\alpha,n,p}(t) = 0$ whenever $\Omega_n r^n \le t < \infty$. Once we show that $u_{r,\alpha,p}$ is decreasing on $[0, \Omega_n r^n)$, the definition of $u_{r,\alpha,n,p}$ implies immediately that $u_{r,\alpha,n,p}$ and $u_{r,\alpha,p}$ have the same distribution function, proving claim (i).

We see that $u_{r,\alpha,p}$ is smooth and strictly positive on $(0, \Omega_n r^n)$. Moreover, it is easy to see that $\lim_{t\to 0} u_{r,\alpha,p}(t) = \infty$. Thus, in order to show that $u_{r,\alpha,p}$ is decreasing on $[0, \Omega_n r^n)$, it is enough to show that $u'_{r,\alpha,p}(t) < 0$ on $(0, \Omega_n r^n)$.

For $t \in (0, \Omega_n r^n)$ we have

$$\begin{aligned} u'_{r,\alpha,p}(t) &= -\frac{1}{p} t^{-1-\frac{1}{p}} \ln^{-\alpha}\left(\frac{\Omega_n r^n e^{p\alpha}}{t}\right) + \alpha t^{-1-\frac{1}{p}} \ln^{-\alpha-1}\left(\frac{\Omega_n r^n e^{p\alpha}}{t}\right) \\ &= t^{-1-\frac{1}{p}} \ln^{-\alpha-1}\left(\frac{\Omega_n r^n e^{p\alpha}}{t}\right)\left(\alpha - \frac{1}{p}\ln\left(\frac{\Omega_n r^n e^{p\alpha}}{t}\right)\right). \end{aligned}$$

We see that

$$\ln\left(\frac{\Omega_n r^n e^{p\alpha}}{t}\right) > 0, \text{ for all } t \in (0, \Omega_n r^n).$$

Thus, for $t \in (0, \Omega_n r^n)$ we have

$$\begin{aligned} u'_{r,\alpha,p}(t) < 0 \quad &\Longleftrightarrow \quad \alpha - \frac{1}{p}\ln\left(\frac{\Omega_n r^n e^{p\alpha}}{t}\right) < 0 \iff \\ \ln\left(\frac{\Omega_n r^n e^{p\alpha}}{t}\right) > p\alpha \quad &\Longleftrightarrow \quad \frac{\Omega_n r^n e^{p\alpha}}{t} > e^{p\alpha} \iff \Omega_n r^n > t. \end{aligned}$$

But the last inequality is obviously true for all $t \in (0, \Omega_n r^n)$. Thus, $u'_{r,\alpha,p}$ is strictly negative on $(0, \Omega_n r^n)$, which implies that $u_{r,\alpha,p}$ is strictly decreasing on $[0, \Omega_n r^n)$.

The definition of $u_{r,\alpha,n,p}$ and the fact that $u_{r,\alpha,p}$ is continuous, strictly decreasing and strictly positive on $(0, \Omega_n r^n)$ imply immediately that $u_{r,\alpha,n,p}$ and $u_{r,\alpha,p}$ have the same distribution function. This yields (5), finishing the proof of (i).

(ii) We proved in part (i) that $u^*_{r,\alpha,n,p}(t) = u^*_{r,\alpha,p}(t)$ for all $t \ge 0$. Thus,

$$||u_{r,\alpha,n,p}||_{L^{p,q}(B(0,r))} = ||u_{r,\alpha,p}||_{L^{p,q}((0,\Omega_n r^n))}$$

for every q in $[1, \infty]$.

For $1 < p < \infty$ and $1 \le q \le \infty$ we let $I_{r,\alpha,p,q} = ||u_{r,\alpha,p}||_{L^{p,q}((0,\Omega_n r^n))}$.

Then via (5) we have

$$\begin{aligned} I_{r,\alpha,p,q} &= \sup_{0\le t\le \Omega_n r^n} t^{\frac{1}{p}} u^*_{r,\alpha,p}(t) = \sup_{0\le t\le \Omega_n r^n} t^{\frac{1}{p}} u_{r,\alpha,p}(t) \\ &= \sup_{0\le t\le \Omega_n r^n} \ln^{-\alpha}\left(\frac{\Omega_n r^n e^{p\alpha}}{t}\right) = (p\alpha)^{-\alpha} \end{aligned}$$

for $q = \infty$ and

$$\begin{aligned} I^q_{r,\alpha,p,q} &= \int_0^{\Omega_n r^n} \left(t^{\frac{1}{p}} u^*_{r,\alpha,p}(t)\right)^q \frac{dt}{t} = \int_0^{\Omega_n r^n} \left(t^{\frac{1}{p}} u_{r,\alpha,p}(t)\right)^q \frac{dt}{t} \\ &= \int_0^{\Omega_n r^n} \ln^{-q\alpha}\left(\frac{\Omega_n r^n e^{p\alpha}}{t}\right) \frac{dt}{t} \end{aligned}$$

for $1 \le q < \infty$.

For a given q in $[1, \infty)$, the last integral in the above sequence is an improper one and converges if and only if $1 - q\alpha < 0$ if and only if $\frac{1}{\alpha} < q$. An easy computation shows that the value of the convergent improper integral is

$$\frac{1}{-1+q\alpha}\ln^{1-q\alpha}\left(\frac{\Omega_n r^n e^{p\alpha}}{\Omega_n r^n}\right) = \frac{(p\alpha)^{1-q\alpha}}{-1+q\alpha}.$$

Thus, if $1 \le q_1 \le \frac{1}{\alpha} < q_2 \le \infty$, we have

$$||u_{r,\alpha,n,p}||_{L^{p,q_2}(B(0,r))} < \infty = ||u_{r,\alpha,n,p}||_{L^{p,q_1}(B(0,r))}.$$

Hence, we proved that $u_{r,\alpha,n,p} \in L^{p,q_2}(B(0,r)) \setminus L^{p,q_1}(B(0,r))$. This shows that the inclusion $L^{p,q_1}(B(0,r)) \subset L^{p,q_2}(B(0,r))$ is strict whenever $1 < p < \infty$ and $1 \le q_1 < q_2 \le \infty$. This finishes the proof of the theorem.

□

Theorem 3.4 allows us to construct a radial function v that is smooth in a punctured ball $B^*(0,r)$ such that $|\nabla v|$ is in $L^{p,q_2}(B(0,r)) \setminus L^{p,q_1}(B(0,r))$. Here $r > 0$, $n \ge 1$, $1 < p < \infty$ and $1 \le q_1 < q_2 \le \infty$.

Theorem 3.5. *Let $n \ge 1$ be an integer. Let $0 < \alpha \le 1$ and $r > 0$. Suppose $1 < p < \infty$ and $1 \le q_1 < q_2 \le \infty$.*

We define

$$f_{rad,r,\alpha,p} : [0,r) \to [0,\infty], f_{rad,r,\alpha,p}(t) = \int_t^r u_{rad,r,\alpha,p}(s)ds, \tag{6}$$

where $u_{rad,r,\alpha,p}$ is the function defined in (3). *We also define*

$$v_{r,\alpha,n,p} : B(0,r) \to [0,\infty], v_{r,\alpha,n,p}(x) := f_{rad,r,\alpha,p}(|x|). \tag{7}$$

Then

(i) $v_{r,\alpha,n,p} \in C^\infty(B^*(0,r))$ *and*

$$\nabla v_{r,\alpha,n,p}(x) = f'_{rad,r,\alpha,p}(|x|)\frac{x}{|x|} \textit{ for all } x \in B^*(0,r).$$

(ii) $|\nabla v_{r,\alpha,n,p}(x)| = u_{r,\alpha,n,p}(x)$ *for all $x \in B^*(0,r)$, where $u_{r,\alpha,n,p}$ is the function defined in* (4).

(iii) $\lim_{x\to y} v_{r,\alpha,n,p}(x) = 0$ *for all $y \in \partial B(0,r)$.*

(iv) *If $p > n$, then $v_{r,\alpha,n,p}$ is continuous in $B(0,r)$.*

(v) *If $1 < p \le n$, then $v_{r,\alpha,n,p}$ is unbounded on $B(0,r)$.*

(vi) $|\nabla v_{r,\alpha,n,p}| \in L^{p,q_2}(B(0,r)) \setminus L^{p,q_1}(B(0,r))$ *if $1 \le q_1 \le \frac{1}{\alpha} < q_2 \le \infty$.*

Proof. Since $u_{rad,r,\alpha,p}$ is smooth in $(0,r)$ and bounded near $t = r$, it follows immediately from the definition of $f_{rad,r,\alpha,p}$ that $f_{rad,r,\alpha,p}$ is smooth in $(0,r)$, $\lim_{t\to r} f_{rad,r,\alpha,p}(t) = 0$ and $f'_{rad,r,\alpha,p}(t) = -u_{rad,r,\alpha,p}(t)$ for all $t \in (0,r)$. This and the definition of $v_{r,\alpha,n,p}$ and $u_{r,\alpha,n,p}$ yield the claims (i), (ii) and (iii) immediately.

Moreover, since

$$\lim_{t\to 0} f'_{rad,r,\alpha,p}(t) = -\lim_{t\to 0} u_{rad,r,\alpha,p}(t) = -\infty,$$

it follows immediately via (i) and (ii) that v is not in $C^\infty(B(0,r))$, because v does not have a gradient at $x = 0 \in B(0,r)$.

We proved in (i) that $v_{r,\alpha,n,p} \in C^\infty(B^*(0,r))$. Thus, the function $v_{r,\alpha,n,p}$ is continuous in $B(0,r)$ if and only if it is continuous at $x = 0 \in B(0,r)$ if and only if $f_{rad,r,\alpha,p}$ is continuous at $t = 0 \in [0,r)$. But from the definition of $f_{rad,r,\alpha,p}$, we see that this function is continuous at $t = 0 \in [0,r)$ if and only $f_{rad,r,\alpha,p}(0) < \infty$. Therefore, $v_{r,\alpha,n,p}$ is continuous in $B(0,r)$ if and only if $f_{rad,r,\alpha,p}(0) < \infty$.

We prove now claim (iv). The definition of $u_{rad,r,\alpha,p}$ easily implies that

$$u_{rad,r,\alpha,p}(s) \le (\Omega_n s^n)^{-\frac{1}{p}} \ln^{-\alpha}(e^{p\alpha}) = (p\alpha)^{-\alpha}(\Omega_n s^n)^{-\frac{1}{p}}$$

for all $s \in (0,r)$.

For $1 \le n < p < \infty$, the definition of $f_{rad,r,\alpha,p}$, the finiteness of the improper Riemann integral $\int_0^r s^{-\frac{n}{p}} ds$, and the Comparison Test for improper Riemann integrals imply immediately that

$$\begin{aligned} f_{rad,r,\alpha,p}(0) &= \lim_{t\to 0} f_{rad,r,\alpha,p}(t) \le (p\alpha)^{-\alpha}\Omega_n^{-\frac{1}{p}} \lim_{t\to 0}\int_t^r s^{-\frac{n}{p}} ds \\ &= (p\alpha)^{-\alpha}\Omega_n^{-\frac{1}{p}}\left(1-\frac{n}{p}\right)^{-1} r^{1-\frac{n}{p}} < \infty. \end{aligned}$$

Thus, if $1 \le n < p < \infty$ we have $f_{rad,r,\alpha,p}(0) < \infty$, which implies (via the above discussion on the boundedness of $f_{rad,r,\alpha,p}(0)$) that $v_{r,\alpha,n,p}$ is continuous in $B(0,r)$.

(v) For $1 < p \le n$, we show that $f_{rad,r,\alpha,p}(0) = \infty$. We treat the cases $1 < p < n$ and $1 < p = n$ separately.

Case I. We consider first the case $1 < p < n$. We begin by showing that there exists a constant $m = m_{r,\alpha,n,p} > 0$ such that

$$t^{\frac{1}{n}-\frac{1}{p}} \ln^{-\alpha}\left(\frac{\Omega_n r^n e^{p\alpha}}{t}\right) \ge m \text{ for all } t \in (0, \Omega_n r^n),$$

which is equivalent to showing that

$$t^{-\frac{1}{p}} \ln^{-\alpha}\left(\frac{\Omega_n r^n e^{p\alpha}}{t}\right) \ge m t^{-\frac{1}{n}} \text{ for all } t \in (0, \Omega_n r^n),$$

which is equivalent to showing that

$$u_{rad,r,\alpha,p}(s) \ge m(\Omega_n s^n)^{-\frac{1}{n}} \text{ for all } s \in (0,r).$$

Once we show the existence of such m, it follows immediately via the Comparison Test for improper Riemann integrals and the definition of $f_{rad,r,\alpha,p}$ that

$$\begin{aligned} f_{rad,r,\alpha,p}(0) &= \lim_{t\to 0} f_{rad,r,\alpha,p}(t) = \lim_{t\to 0}\int_t^r u_{rad,r,\alpha,p}(s) ds \\ &\ge m\Omega_n^{-\frac{1}{n}} \lim_{t\to 0}\int_t^r s^{-1} ds = \infty. \end{aligned}$$

This would prove the unboundedness of $v_{r,\alpha,n,p}$ on $B(0,r)$ when $1 < p < n$.

We let $p_1 = \frac{np}{n-p}$. Thus, $p_1 > p$ and $\frac{1}{p_1} = \frac{1}{p} - \frac{1}{n}$. We define h on the interval $[0, \Omega_n r^n)$ by

$$h(t) = h_{r,\alpha,n,p}(t) = t^{-\frac{1}{p_1}} \ln^{-\alpha}\left(\frac{\Omega_n r^n e^{p\alpha}}{t}\right). \tag{8}$$

We notice that h is smooth and strictly positive on $(0, \Omega_n r^n)$. Moreover, it is easy to see that $\lim_{t\to 0} h(t) = \infty$. We compute h' on $(0, \Omega_n r^n)$ and we notice that

$$h'(t) = t^{-1-\frac{1}{p_1}} \ln^{-\alpha-1}\left(\frac{\Omega_n r^n e^{p\alpha}}{t}\right)\left(\alpha - \frac{1}{p_1}\ln\left(\frac{\Omega_n r^n e^{p\alpha}}{t}\right)\right), t \in (0, \Omega_n r^n).$$

We see that $h'(t) = 0$ if and only if $t = t_{crit} = \Omega_n r^n e^{p\alpha - p_1\alpha} \in (0, \Omega_n r^n)$. We notice that h has an unique global minimum on $(0, \Omega_n r^n)$, at $t = t_{crit}$. We define $m = m_{r,\alpha,n,p} := h(t_{crit})$. Then $m > 0$ and $h(t) \ge m > 0$ for all $t \in [0, \Omega_n r^n)$. This proves the existence of the desired constant m and finishes the proof of Case I.

Case II. We consider now the case $1 < p = n$.

We compute effectively $f_{rad,r,\alpha,n}$ by considering the cases $\alpha = 1$ and $\alpha \in (0,1)$ separately.

We assume first that $\alpha = 1$. For every $t \in (0, r)$ we have

$$\begin{aligned} f_{rad,r,1,n}(t) &= \int_t^r (\Omega_n s^n)^{-\frac{1}{n}} \ln^{-1}\left(\frac{\Omega_n r^n e^n}{\Omega_n s^n}\right) ds \\ &= \Omega_n^{-\frac{1}{n}} \int_t^r s^{-1} \ln^{-1}\left(\frac{r^n e^n}{s^n}\right) ds \\ &= \Omega_n^{-\frac{1}{n}} n^{-1} \int_t^r s^{-1} \ln^{-1}\left(\frac{re}{s}\right) ds \\ &= \Omega_n^{-\frac{1}{n}} n^{-1} \ln\left(\ln\left(\frac{re}{t}\right)\right) = \Omega_n^{-\frac{1}{n}} n^{-1} \ln\left(1 + \ln\left(\frac{r}{t}\right)\right). \end{aligned}$$

Thus,

$$v_{r,1,n,n}(x) = \Omega_n^{-\frac{1}{n}} n^{-1} \ln\left(1 + \ln\left(\frac{r}{|x|}\right)\right) \text{ for all } x \in B(0, r).$$

It is easy to see that $v_{r,1,n,n}$ is unbounded on $B(0, r)$. This proves Case II when $1 < p = n$ and $\alpha = 1$.

We assume now that $\alpha \in (0, 1)$. For every $t \in (0, r)$ we have

$$\begin{aligned} f_{rad,r,\alpha,n}(t) &= \int_t^r (\Omega_n s^n)^{-\frac{1}{n}} \ln^{-\alpha}\left(\frac{\Omega_n r^n e^{n\alpha}}{\Omega_n s^n}\right) ds \\ &= \Omega_n^{-\frac{1}{n}} \int_t^r s^{-1} \ln^{-\alpha}\left(\frac{r^n e^{n\alpha}}{s^n}\right) ds \\ &= \Omega_n^{-\frac{1}{n}} n^{-\alpha} \int_t^r s^{-1} \ln^{-\alpha}\left(\frac{re^{\alpha}}{s}\right) ds \\ &= \Omega_n^{-\frac{1}{n}} n^{-\alpha} (1-\alpha)^{-1} \left(\ln^{1-\alpha}\left(\frac{re^{\alpha}}{t}\right) - \alpha^{1-\alpha}\right). \end{aligned}$$

Thus,

$$v_{r,\alpha,n,n}(x) = \frac{1}{\Omega_n^{\frac{1}{n}} n^{\alpha}(1-\alpha)} \left(\ln^{1-\alpha}\left(\frac{re^{\alpha}}{|x|}\right) - \alpha^{1-\alpha}\right) \text{ for all } x \in B(0, r).$$

It is easy to see that $v_{r,\alpha,n,n}$ is unbounded on $B(0, r)$. This proves Case II when $1 < p = n$ and $\alpha \in (0, 1)$. This finishes the proof of claim (v).

We prove now (vi). From part (ii) we have $|\nabla v_{r,\alpha,n,p}(x)| = u_{r,\alpha,n,p}(x)$ for all $x \in B^*(0, r)$. The claim follows immediately from the choice of α and Theorem 3.4 (ii). This finishes the proof of the theorem.

□

The following proposition shows that $L^{p,\infty}$ does not have an absolutely continuous (p, ∞)-norm. Moreover, it exhibits a function $u \in L^{p,\infty}$ that does not have absolutely continuous (p, ∞)-norm and is not in $L^{p,q}$ for any q in $[1, \infty)$.

Proposition 3.6. *Let $n \geq 1$ be an integer. Let $r > 0$ and $1 < p < \infty$. We define*

$$u_r : B(0, r) \to [0, \infty], u_r(x) = |x|^{-\frac{n}{p}}, 0 \leq |x| < r.$$

Then

(i) $u_r \in L^{p,\infty}(B(0, r))$ *and* $||u_r||_{L^{p,\infty}(B(0,r))} = \Omega_n^{\frac{1}{p}}$.
(ii) $u_r \notin L^{p,q}(B(0, r))$ *for every* $q \in [1, \infty)$.
(iii) u_r *does not have absolutely continuous* (p, ∞)*-norm.*
(iv) *If* $v : B(0, r) \to \mathbf{R}$ *is a locally bounded Lebesgue measurable function on* $B(0, r)$*, then*

$$||u_r - v||_{L^{p,\infty}(B(0,\alpha))} \geq ||u_r||_{L^{p,\infty}(B(0,r))}$$

for every $\alpha \in (0, r)$.

Proof. We compute u_r^*, the nonincreasing rearrangement of u_r. In order to do that, we first compute $\lambda_{[u_r]}$, the distribution function of u_r. For every $t \in [0, \infty)$ we have

$$\begin{aligned}\lambda_{[u_r]}(t) &= |\{x \in B(0,r) : |u_r(x)| > t\}| = |\{x \in B(0,r) : |x|^{-\frac{n}{p}} > t\}| \\ &= |\{x \in B(0,r) : |x| < t^{-\frac{p}{n}}\}| = |B(0, t^{-\frac{p}{n}}) \cap B(0,r)| \\ &= \min(\Omega_n t^{-p}, \Omega_n r^n).\end{aligned}$$

Thus,

$$u_r^*(t) = \begin{cases} \left(\frac{\Omega_n}{t}\right)^{\frac{1}{p}} & \text{if } t \in [0, \Omega_n r^n) \\ 0 & \text{if } t \in [\Omega_n r^n, \infty). \end{cases}$$

This implies immediately that

$$||u_r||_{L^{p,\infty}(B(0,r))} = \sup_{t \in [0, \Omega_n r^n)} t^{\frac{1}{p}} u_r^*(t) = \sup_{t \in [0, \Omega_n r^n]} t^{\frac{1}{p}} (\Omega_n t^{-1})^{\frac{1}{p}} = \Omega_n^{\frac{1}{p}}$$

and

$$\begin{aligned}||u_r||^q_{L^{p,q}(B(0,r))} &= \int_0^{\Omega_n r^n} (t^{\frac{1}{p}} u_r^*(t))^q \frac{dt}{t} \\ &= \int_0^{\Omega_n r^n} \left(t^{\frac{1}{p}} (\Omega_n t^{-1})^{\frac{1}{p}}\right)^q \frac{dt}{t} \\ &= \int_0^{\Omega_n r^n} \Omega_n^{\frac{q}{p}} \frac{dt}{t} = \infty\end{aligned}$$

for all q in $[1, \infty)$. This proves (i) and (ii).

(iii) We prove now that the function u_r does not have an absolutely continuous (p, ∞)-norm. Let $\alpha \in (0, r)$ be fixed. Let $u_{r,\alpha} : B(0,r) \to [0, \infty]$ be the restriction of u_r to $B(0, \alpha)$. By doing a computation very similar to the computation of u_r, we have

$$u_{r,\alpha}^*(t) = \begin{cases} \left(\frac{\Omega_n}{t}\right)^{\frac{1}{p}} & \text{if } t \in [0, \Omega_n \alpha^n) \\ 0 & \text{if } t \in [\Omega_n \alpha^n, \infty). \end{cases}$$

Thus,

$$||u_{r,\alpha}||_{L^{p,\infty}(B(0,\alpha))} = ||u_r||_{L^{p,\infty}(B(0,\alpha))} = ||u_r||_{L^{p,\infty}(B(0,r))} = \Omega_n^{\frac{1}{p}} \tag{9}$$

for every $\alpha \in (0, r)$. This shows that u_r does not have an absolutely continuous (p, ∞)-norm. This proves (iii).

We prove now (iv). Let $v : B(0,r) \to \mathbf{R}$ be a Lebesgue measurable function that is locally bounded on $B(0,r)$. (Any continuous function on $B(0,r)$ is such a function). Let $\alpha \in (0, r)$ and $\varepsilon \in (0,1)$ be fixed. Let $M_\alpha > 0$ be chosen such that $|v(x)| < M_\alpha$ for all $x \in B(0, \alpha)$. We have

$$|u_r(x) - v(x)| \geq |u_r(x)| - |v(x)| \geq |u_r(x)| - M_\alpha$$

for all $x \in B(0, \alpha)$. We want to find $\alpha_\varepsilon \in (0, \alpha)$ such that $M_\alpha < \varepsilon |u_r(x)|$ for all $x \in B(0, \alpha_\varepsilon)$. We have

$$M_\alpha < \varepsilon |u_r(x)| \iff \frac{M_\alpha}{\varepsilon} < |x|^{-\frac{n}{p}} \iff |x|^{\frac{n}{p}} < \frac{\varepsilon}{M_\alpha} \iff |x| < \left(\frac{\varepsilon}{M_\alpha}\right)^{\frac{p}{n}}.$$

If we choose

$$\alpha_\varepsilon = \min\left(\alpha, \left(\frac{\varepsilon}{M_\alpha}\right)^{\frac{p}{n}}\right),$$

the above computation, the definition of u_r and the fact that $|v| < M_\alpha$ on $B(0,\alpha)$ imply that

$$|u_r(x) - v(x)| \geq (1-\varepsilon)|u_r(x)|$$

for all x in $B(0,\alpha_\varepsilon)$. Thus, we have

$$\begin{aligned} ||u_r - v||_{L^{p,\infty}(B(0,\alpha))} &\geq ||u_r - v||_{L^{p,\infty}(B(0,\alpha_\varepsilon))} \geq (1-\varepsilon)||u_r||_{L^{p,\infty}(B(0,\alpha_\varepsilon))} \\ &= (1-\varepsilon)||u_r||_{L^{p,\infty}(B(0,r))}. \end{aligned}$$

The inequalities in the above sequence are obvious; we use (9) for the equality in the above sequence. By letting $\varepsilon \to 0$, we obtain the desired conclusion for a fixed $\alpha \in (0,r)$. Thus, we proved claim (iv). This finishes the proof.

□

3.4. **Hölder Inequalities for Lorentz Spaces.** Here we record the following generalized Hölder inequalities for Lorentz spaces, previously proved in [7] and/or in [9], valid for all integers $n \geq 1$.

Theorem 3.7. (See Costea [7, Theorem 2.3] and [9, Theorem 2.2.1]). *Let $\Omega \subset \mathbf{R}^n$. Suppose $1 < p < \infty$ and $1 \leq q \leq \infty$. If $f \in L^{p,q}(\Omega)$ and $g \in L^{p',q'}(\Omega)$, then*

$$\int_\Omega |f(x)g(x)|dx \leq \int_0^\infty f^*(s)g^*(s)ds \leq ||f||_{L^{p,q}(\Omega)}||g||_{L^{p',q'}(\Omega)}.$$

We have the following generalized Hölder inequality for Lorentz spaces, valid for all integers $n \geq 1$.

Theorem 3.8. (See Costea [9, Theorem 2.2.2]). *Suppose $\Omega \subset \mathbf{R}^n$ has finite measure. Let $1 < p_1, p_2, p_3 < \infty$, $1 \leq q_1, q_2, q_3 \leq \infty$ be such that*

$$\frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p_3}$$

and either

$$\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3}$$

whenever $1 \leq q_1, q_2, q_3 < \infty$ or $1 \leq q_1 = q_2 \leq q_3 = \infty$ or $1 \leq q_1 = q_3 \leq q_2 = \infty$. Then

$$||f||_{L^{p_1,q_1}(\Omega;\mathbf{R}^m)} \leq ||f||_{L^{p_2,q_2}(\Omega;\mathbf{R}^m)}\,||\chi_\Omega||_{L^{p_3,q_3}(\Omega)}.$$

As an application of Theorem 3.8 we have the following result, valid for all integers $n \geq 1$.

Corollary 3.9. (See Costea [7, Corollary 2.4] and [9, Corollary 2.2.3]). *Let $1 < p < q \leq \infty$ and $\varepsilon \in (0, p-1)$ be fixed. Suppose $\Omega \subset \mathbf{R}^n$ has finite measure. Then*

$$||f||_{L^{p-\varepsilon}(\Omega;\mathbf{R}^m)} \leq C(p,q,\varepsilon)\,|\Omega|^{\frac{\varepsilon}{p(p-\varepsilon)}}\,||f||_{L^{p,q}(\Omega;\mathbf{R}^m)} \tag{10}$$

for every integer $m \geq 1$, where

$$C(p,q,\varepsilon) = \begin{cases} \left(\frac{p(q-p+\varepsilon)}{q}\right)^{\frac{1}{p-\varepsilon}-\frac{1}{q}} \varepsilon^{\frac{1}{q}-\frac{1}{p-\varepsilon}}, & p < q < \infty \\ p^{\frac{1}{p-\varepsilon}}\,\varepsilon^{-\frac{1}{p-\varepsilon}}, & q = \infty. \end{cases}$$

For the following definition, see Bennett-Sharpley [1, Definition IV.4.17].

Definition 3.10. For every measurable function f on $\mathbf{R}^n$, $n \geq 2$, the *fractional integral* $I_1 f$ is defined by

$$(I_1 f)(x) = \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-1}}\,dy.$$

We record here the Hardy-Littlewood-Sobolev theorem of fractional integration. (See Bennett-Sharpley [1, Theorem IV.4.18] and Costea [9, Theorem 2.2.5]).

Theorem 3.11. Hardy-Littlewood-Sobolev theorem. *Let $1 < p < n$ and $1 \le q \le \infty$. Then there exists a constant $C(n,p,q) > 0$ such that*

$$||I_1 f||_{L^{\frac{np}{n-p},q}(\mathbf{R}^n)} \le C(n,p,q)||f||_{L^{p,q}(\mathbf{R}^n)} \tag{11}$$

whenever $f \in L^{p,q}(\mathbf{R}^n)$.

4. Sobolev-Lorentz Spaces

This section is based in part on Chapter V of my PhD thesis [6] and on Chapter 3 of my book [9]. We generalize and extend some of the results from [6] and [9] to the case $n = 1$.

Among the new results in this section we mention the case $q = \infty$ for Theorems 4.11 and 4.12 as well as the inclusion $W^{1,(p,q)}(\Omega) \subsetneq W^{1,(p,s)}(\Omega)$, where $\Omega \subset \mathbf{R}^n$ is open, $n \ge 1$ is an integer, $1 < p < \infty$ and $1 \le q < s \le \infty$.

4.1. **The $H^{1,(p,q)}$ and $W^{1,(p,q)}$ Spaces.** For $1 < p < \infty$ and $1 \le q \le \infty$ we define the Sobolev-Lorentz space $H^{1,(p,q)}(\Omega)$ as follows. Let $r = \min(p,q)$. For a function $\phi \in C^\infty(\Omega)$ we define its Sobolev-Lorentz (p,q)-norm by

$$||\phi||_{1,(p,q);\Omega} = \left(||\phi||^r_{L^{(p,q)}(\Omega)} + ||\nabla\phi||^r_{L^{(p,q)}(\Omega;\mathbf{R}^n)}\right)^{\frac{1}{r}},$$

where, we recall, $\nabla\phi = (\partial_1\phi, \ldots, \partial_n\phi)$ is the gradient of ϕ. Similarly we define the Sobolev-Lorentz p,q-quasinorm of ϕ by

$$||\phi||_{1,p,q;\Omega} = \left(||\phi||^r_{L^{p,q}(\Omega)} + ||\nabla\phi||^r_{L^{p,q}(\Omega;\mathbf{R}^n)}\right)^{\frac{1}{r}},$$

Then $H^{1,(p,q)}(\Omega)$ is defined as the completion of

$$\{\phi \in C^\infty(\Omega) : ||\phi||_{1,(p,q);\Omega} < \infty\}$$

with respect to the norm $||\cdot||_{1,(p,q);\Omega}$. Throughout the paper we might also use $||\cdot||_{H^{1,(p,q)}(\Omega)}$ instead of $||\cdot||_{1,(p,q);\Omega}$ and $||\cdot||_{H^{1,p,q}(\Omega)}$ instead of $||\cdot||_{1,p,q;\Omega}$.

The Sobolev-Lorentz space $H_0^{1,(p,q)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $H^{1,(p,q)}(\Omega)$. The Sobolev-Lorentz spaces $H_0^{1,(p,q)}(\Omega)$ and $H^{1,(p,q)}(\Omega)$ can be both regarded as closed subspaces of $L^{(p,q)}(\Omega) \times L^{(p,q)}(\Omega;\mathbf{R}^n)$. Since $L^{(p,q)}(\Omega) \times L^{(p,q)}(\Omega;\mathbf{R}^n)$ is reflexive when $1 < q < \infty$, it follows that both $H_0^{1,(p,q)}(\Omega)$ and $H^{1,(p,q)}(\Omega)$ are reflexive Banach spaces when $1 < q < \infty$ and have absolutely continuous norm when $1 \le q < \infty$. In particular, $u \in L^{(p,q)}(\Omega)$ and $\nabla u \in L^{(p,q)}(\Omega;\mathbf{R}^n)$ have absolutely continuous (p,q)-norm whenever $1 < p < \infty$ and $1 \le q < \infty$.

For $q = 1$ we have that $(L^{(p,1)}(\Omega))^* \times (L^{(p,1)}(\Omega;\mathbf{R}^n))^*$ can be regarded as a subspace of $(H^{1,(p,1)}(\Omega))^*$ and since $(L^{(p,1)}(\Omega))^* \times (L^{(p,1)}(\Omega;\mathbf{R}^n))^*$ can be identified with the non-reflexive space $L^{(p',\infty)}(\Omega) \times L^{(p',\infty)}(\Omega;\mathbf{R}^n)$, it follows that $H^{1,(p,1)}(\Omega)$ is non-reflexive and so is $H_0^{1,(p,1)}(\Omega)$, since it is a closed subspace of $H^{1,(p,1)}(\Omega)$. It will be proved later in Theorem 4.8 that none of these two spaces is reflexive when $q = \infty$.

Next we record the following reflexivity result, valid for all $n \ge 1$ and for all $q \in (1,\infty)$.

Theorem 4.1. (See Costea [6, Theorem V.22] and [9, Theorem 3.5.4]). Let $1 < p, q < \infty$. Suppose that u_j is a bounded sequence in $H^{1,(p,q)}(\Omega)$ such that $u_j \to u$ pointwise almost everywhere in Ω. Then $u \in H^{1,(p,q)}(\Omega)$. Moreover, if $u_j \in H_0^{1,(p,q)}(\Omega)$ for all $j \ge 1$, then $u \in H_0^{1,(p,q)}(\Omega)$.

The following theorem generalizes the Gagliardo-Nirenberg-Sobolev inequality to the Sobolev-Lorentz spaces $H_0^{1,(p,q)}(\Omega)$ for $1 < p < n$ and $1 \le q \le \infty$. It also presents a Sobolev-Poincaré inequality for the Sobolev-Lorentz spaces $H_0^{1,(p,q)}(\Omega)$ when $\Omega \subset \mathbf{R}^n$ is open and bounded, $1 < p < \infty$ and $1 \le q \le \infty$.

Theorem 4.2. Sobolev inequalities for Sobolev-Lorentz spaces.

Let $\Omega \subset \mathbf{R}^n$ be an open set, where $n \geq 2$ is an integer. Suppose $1 < p < \infty$ and $1 \leq q \leq \infty$.

(i) *If $1 < p < n$, then there exists a constant $C(n, p, q) > 0$ such that*

$$||u||_{L^{\frac{np}{n-p},q}(\Omega)} \leq C(n,p,q)||\nabla u||_{L^{p,q}(\Omega;\mathbf{R}^n)}$$

for every $u \in H_0^{1,(p,q)}(\Omega)$.

(ii) (See Costea [9, Theorem 3.1.1]). *If Ω is bounded, then there exists a constant $C(n, p, q) > 0$ such that*

$$||u||_{L^{p,q}(\Omega)} \leq C(n,p,q)\,|\Omega|^{\frac{1}{n}}||\nabla u||_{L^{p,q}(\Omega;\mathbf{R}^n)} \tag{12}$$

for every $u \in H_0^{1,(p,q)}(\Omega)$.

Proof. We have that $H_0^{1,(p,q)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^{1,(p,q)}(\Omega)$. Thus, via Costea [7, Corollary 2.7], it is enough to prove claims (i) and (ii) for functions $u \in C_0^\infty(\Omega)$.

Let u be in $C_0^\infty(\Omega)$. We extend the function u by 0 on $\mathbf{R}^n \setminus \Omega$ and we denote this extension by u as well. Then u is in $C_0^\infty(\mathbf{R}^n)$ and u is compactly supported in Ω. Via Gilbarg-Trudinger [15, Lemma 7.14], we have

$$|u(x)| \leq \frac{1}{\omega_{n-1}}(I_1|\nabla u|)(x)$$

for every $x \in \mathbf{R}^n$. By using this pointwise inequality together with the Hardy-Littlewood-Sobolev Theorem (see Theorem 3.11 and Bennett-Sharpley [1, Theorem IV.4.18]) it follows immediately that claim (i) holds for all functions $u \in C_0^\infty(\Omega)$. This proves claim (i) via Costea [7, Corollary 2.7].

We prove now claim (ii). We have to consider two cases, depending on whether $1 < p < n$ or $n \leq p < \infty$.

Case I. First we assume that $1 < p < n$. We notice that $p < \frac{np}{n-p}$. Via Theorems 3.8 and 3.11 it follows from part (i) that

$$||u||_{L^{p,q}(\Omega)} \leq |\Omega|^{\frac{1}{n}}||u||_{L^{\frac{np}{n-p},q}(\Omega)} \leq \frac{C(n,p,q)}{\omega_{n-1}}|\Omega|^{\frac{1}{n}}||\nabla u||_{L^{p,q}(\Omega;\mathbf{R}^n)}$$

for every $u \in C_0^\infty(\Omega)$, where $C(n, p, q)$ is the constant from Theorem 3.11. This proves the claim (ii) for $1 < p < n$ via Costea [7, Corollary 2.7].

Case II. We assume now that $1 < n \leq p < \infty$. We choose $s \in (1, n)$ such that $p < \frac{ns}{n-s}$. Via Theorems 3.8 and 3.11 it follows from part (i) that

$$\begin{aligned} ||u||_{L^{p,q}(\Omega)} &\leq |\Omega|^{\frac{1}{p}-\frac{n-s}{ns}}||u||_{L^{\frac{ns}{n-s},q}(\Omega)} \\ &\leq \frac{C(n,s,q)}{\omega_{n-1}}|\Omega|^{\frac{1}{p}-\frac{n-s}{ns}}||\nabla u||_{L^{s,q}(\Omega;\mathbf{R}^n)} \\ &\leq \frac{C(n,s,q)}{\omega_{n-1}}|\Omega|^{\frac{1}{n}}||\nabla u||_{L^{p,q}(\Omega;\mathbf{R}^n)} \end{aligned}$$

for every $u \in C_0^\infty(\Omega)$, where $C(n, s, q)$ is the constant from Theorem 3.11. This proves the claim (ii) for $1 < n \leq p < \infty$ via Costea [7, Corollary 2.7]. This finishes the proof of the theorem. □

We recall that for $1 < p < \infty$, $H^{1,p}(\Omega)$ is defined as the closure of $C^\infty(\Omega)$ with respect to the $||\cdot||_{1,p;\Omega}$-norm, where

$$||\psi||_{1,p;\Omega} = \left(\int_\Omega |\psi(x)|^p dx + \int_\Omega |\nabla\psi(x)|^p dx\right)^{\frac{1}{p}}$$

for every $\psi \in C^\infty(\Omega)$. We recall that $H^{1,p}_{loc}(\Omega)$ is defined in the obvious manner: a measurable function $u : \Omega \to \mathbf{R}$ is in $H^{1,p}_{loc}(\Omega)$ if and only if u is in $H^{1,p}_{loc}(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.

Let $u \in L^1_{loc}(\Omega)$. For $i = 1, \ldots, n$ a function $v \in L^1_{loc}(\Omega)$ is called the *ith weak partial derivative of* u and we denote $v = \partial_i u$ if

$$\int_\Omega \varphi\, v\, dx = -\int_\Omega \partial_i \varphi\, u\, dx$$

for all $\varphi \in C_0^\infty(\Omega)$. Recall that

$$W^{1,p}(\Omega) = L^p(\Omega) \cap \{u : \partial_i u \in L^p(\Omega),\, i = 1, \ldots, n\}.$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$||u||_{W^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + \sum_{i=1}^n ||\partial_i u||_{L^p(\Omega)},$$

which is clearly equivalent to

$$\left(||u||^p_{L^p(\Omega)} + ||\nabla u||^p_{L^p(\Omega;\mathbf{R}^n)}\right)^{\frac{1}{p}}.$$

Here ∇u is the distributional gradient of u. We recall that $W^{1,p}(\Omega) = H^{1,p}(\Omega)$.

We define the Sobolev-Lorentz space $W^{1,(p,q)}(\Omega)$ by

$$W^{1,(p,q)}(\Omega) = L^{(p,q)}(\Omega) \cap \{u : \partial_i u \in L^{(p,q)}(\Omega),\, i = 1, \ldots, n\}.$$

The space $W^{1,(p,q)}(\Omega)$ is equipped with the norm

$$||u||_{W^{1,(p,q)}(\Omega)} = ||u||_{L^{(p,q)}(\Omega)} + \sum_{i=1}^n ||\partial_i u||_{L^{(p,q)}(\Omega)},$$

which is clearly equivalent to

$$\left(||u||^r_{L^{(p,q)}(\Omega)} + ||\nabla u||^r_{L^{(p,q)}(\Omega;\mathbf{R}^n)}\right)^{\frac{1}{r}},$$

where $r = \min(p, q)$. As earlier, it is easy to see that $W^{1,(p,q)}(\Omega)$ is a reflexive Banach space when $1 < q < \infty$ and a non-reflexive Banach space when $q = 1$. It will be proved later in Theorem 4.8 that $W^{1,(p,\infty)}(\Omega)$ is not reflexive.

The corresponding local space $H^{1,(p,q)}_{loc}(\Omega)$ is defined in the obvious manner: u is in $H^{1,(p,q)}_{loc}(\Omega)$ if and only if u is in $H^{1,(p,q)}(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.

Similarly, the local space $W^{1,(p,q)}_{loc}(\Omega)$ is defined as follows: u is in $W^{1,(p,q)}_{loc}(\Omega)$ if and only if u is in $W^{1,(p,q)}(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.

The following theorem shows, among other things, the relation between $W^{1,(p,q)}(\Omega)$ and $H^{1,s}_{loc}(\Omega)$, where $1 < s < p < \infty$ and $1 \le q \le \infty$.

Theorem 4.3. *Let $\Omega \subset \mathbf{R}^n$ be an open set, where $n \ge 1$ is an integer. Let $1 < s < p < \infty$ and $1 \le q < r \le \infty$.*

(i) *We have $W^{1,(p,q)}(\Omega) \subset H^{1,s}_{loc}(\Omega)$. Moreover, if Ω has finite Lebesgue measure (in particular if Ω is bounded), then $W^{1,(p,q)}(\Omega) \subset H^{1,s}(\Omega)$.*

(ii) *If Ω is bounded, then $H^{1,(p,q)}_0(\Omega) \subset H^{1,s}_0(\Omega)$.*

(iii) *We have $H^{1,(p,q)}_0(\Omega) \subset H^{1,(p,r)}_0(\Omega)$, $H^{1,(p,q)}(\Omega) \subset H^{1,(p,r)}(\Omega)$, and $W^{1,(p,q)}(\Omega) \subset W^{1,(p,r)}(\Omega)$.*

Proof. For claim (i), see Costea [6, Theorem V.2] and [9, Theorem 3.1.2]. From either of these two references we can copy almost verbatim the proof, valid also for $n = 1$.

Claims (ii) and (iii) follow immediately from Remark 3.3, Corollary 3.9, and the definition of the Sobolev-Lorentz spaces on Ω.

□

We record the following theorem, which shows that every Sobolev element u in $H^{1,(p,q)}_{loc}(\Omega)$ is a distribution.

Theorem 4.4. (See Costea [6, Theorem V.3] and [9, Theorem 3.1.3]). *Suppose* $1 < p < \infty$ *and* $1 \leq q \leq \infty$. *Let* u *be in* $H^{1,(p,q)}_{loc}(\Omega)$. *Then* u *is a distribution with distributional gradient* $\nabla u \in L^1_{loc}(\Omega; \mathbf{R}^n)$. *Moreover,* $u \in L^{(p,q)}_{loc}(\Omega) \subset L^1_{loc}(\Omega)$ *and*

$$\int_\Omega u\, \partial_i \varphi\, dx = - \int_\Omega \partial_i u\, \varphi\, dx$$

for all $\varphi \in C_0^\infty(\Omega)$ *and* $i = 1, \ldots, n$, *where* $\partial_i u$ *is the* i*th coordinate of* ∇u. *In particular,* $H^{1,(p,q)}(\Omega) \subset W^{1,(p,q)}(\Omega)$.

4.2. **Regularization.** We need some basic properties of the Sobolev-Lorentz spaces. Before proceeding we recall the usual regularization procedure.

Let $\eta \in C_0^\infty(B(0,1))$ be a *mollifier*. This means that η is a nonnegative function such that

$$\int_{\mathbf{R}^n} \eta(x)\, dx = 1.$$

Without loss of generality we can assume that η is a radial function. Next we write

$$\eta_\varepsilon(x) = \varepsilon^{-n} \eta(\varepsilon^{-1} x),\ \varepsilon > 0.$$

For the basic properties of a mollifier see Ziemer [30, Theorems 1.6.1 and 2.1.3]. We summarize the properties of the convolution (valid for all integers $n \geq 1$) in the following theorem.

Theorem 4.5. (See Costea [6, Theorem V.4] and [9, Theorem 3.2.1]). *For* $v \in L^1_{loc}(\mathbf{R}^n)$, *the convolution*

$$v_\varepsilon(x) = \eta_\varepsilon * v(x) = \int_{\mathbf{R}^n} \eta_\varepsilon(x-y) v(y) dy$$

enjoys the following properties for every $\varepsilon > 0$*:*

(i) *For every* $p \in (1, \infty)$ *and every* $q \in [1, \infty]$, *there exists a constant* $C(p,q) > 0$ *such that*

$$||v_\varepsilon||_{L^{(p,q)}(\mathbf{R}^n)} \leq C(p,q) ||v||_{L^{(p,q)}(\mathbf{R}^n)}. \tag{13}$$

(ii) *For every* $p \in (1, \infty)$, *every* $q \in [1, \infty]$ *and every* $v \in L^{(p,q)}(\mathbf{R}^n)$ *with absolutely continuous* (p,q)*-norm, we have*

$$||v_\varepsilon - v||_{L^{(p,q)}(\mathbf{R}^n)} \to 0 \tag{14}$$

as $\varepsilon \to 0$.

Recall that a function $u : \Omega \to \mathbf{R}$ is *Lipschitz* on $\Omega \subset \mathbf{R}^n$, if there is $L > 0$ such that

$$|u(x) - u(y)| \leq L|x-y|$$

for all $x, y \in \Omega$. Moreover, u is *locally Lipschitz* on Ω if u is Lipshitz on each compact subset of Ω.

It is well known that every locally Lipschitz function on $\mathbf{R}^n$ is differentiable; this is Rademacher's theorem (see Federer [13, Theorem 3.1.6]).

Lemma 4.6. (See Costea [6, Lemma V.5] and [9, Theorem 3.2.2]). *Suppose* $1 < p < \infty$ *and* $1 \leq q \leq \infty$. *Let* $u : \Omega \to \mathbf{R}$ *be a locally Lipschitz function. Then* $u \in H^{1,(p,q)}_{loc}(\Omega)$ *and* $\nabla u = (\partial_1 u, \ldots, \partial_n u)$ *is the usual gradient of* u.

4.3. **Product rule, density results and strict inclusions for Sobolev-Lorentz spaces.**

Proposition 4.7. *Let* $n \geq 1$ *be an integer and let* $1 < p < \infty$. *Let* $u_p : \mathbf{R}^n \to [-\infty, \infty]$,

$$u_p(x) = \begin{cases} \ln|x| & \text{if } p = n > 1 \\ |x|^{1-\frac{n}{p}} & \text{if } p \neq n. \end{cases}$$

Then u_p *is in* $W^{1,(p,\infty)}_{loc}(\mathbf{R}^n) \setminus H^{1,(p,\infty)}_{loc}(\mathbf{R}^n)$.

Proof. Since from Theorem 4.3 we have $W^{1,(p,\infty)}_{loc}(\mathbf{R}^n) \subset H^{1,s}_{loc}(\mathbf{R}^n)$ for all $1 < s < p$, we prove first that u_p is in $H^{1,s}_{loc}(\mathbf{R}^n)$ for all $1 < s < p$.

We start by noticing that $u_p \in L^p_{loc}(\mathbf{R}^n)$ for $p = n > 1$ and that $u_p(x) \le r|x|^{-\frac{n}{p}}$ for all $x \in B(0,r)$ and for all $p \ne n$. Thus, u_p is in $L^{(p,\infty)}_{loc}(\mathbf{R}^n)$ for all $p \in (1,\infty)$. Moreover, an easy computation shows that

$$\lim_{r\to\infty} ||u_p||_{L^{p,\infty}(B(0,r))} = \infty$$

for all $p \in (1,\infty)$. Thus, u_p is not in $L^{p,\infty}(\mathbf{R}^n)$.

We notice that u_p is smooth in $\mathbf{R}^n \setminus \{0\}$ with

$$\nabla u_p(x) = C(n,p)\, x\, |x|^{-1-\frac{n}{p}}, x \ne 0,$$

where

$$C(n,p) = \begin{cases} 1 & \text{if } p = n > 1 \\ 1 - \frac{n}{p} & \text{if } p \ne n. \end{cases} \tag{15}$$

Thus, $|\nabla u_p(x)| = |C(n,p)|\, |x|^{-\frac{n}{p}}, x \ne 0$. By doing a computation similar to the one in Proposition 3.6, we have

$$|\nabla u_p|^*(t) = |C(n,p)| \left(\frac{\Omega_n}{t}\right)^{1/p}$$

for all $t \ge 0$, where $C(n,p)$ is the constant from (15).

Thus, it follows immediately that $|\nabla u_p|$ is in $L^{(p,\infty)}(\mathbf{R}^n)$ and

$$||\nabla u_p||_{L^{p,\infty}(\mathbf{R}^n;\mathbf{R}^n)} = ||\nabla u_p||_{L^{p,\infty}(B(0,r);\mathbf{R}^n)} = |C(n,p)|\, \Omega_n^{1/p} < ||\nabla u_p||_{L^{p,q}(B(0,r);\mathbf{R}^n)} = \infty$$

for every $r > 0$ and every $1 \le q < \infty$, where $C(n,p)$ is the above constant.

By invoking Proposition 3.6 (iv), we see that

$$||\nabla u_p - \nabla v||_{L^{p,\infty}(B(0,\alpha);\mathbf{R}^n)} \ge ||\nabla u_p||_{L^{p,\infty}(B(0,r);\mathbf{R}^n)} = ||\nabla u_p||_{L^{p,\infty}(\mathbf{R}^n;\mathbf{R}^n)} = |C(n,p)|\, \Omega_n^{1/p} > 0$$

for every $v \in C^\infty(\mathbf{R}^n)$ and every $0 < \alpha < r < \infty$, where $C(n,p)$ is the constant from (15). This implies immediately that u_p is not in $H^{1,(p,\infty)}_{loc}(\mathbf{R}^n)$ because u_p cannot be approximated with smooth functions in the $H^{1,(p,\infty)}$ norm on open balls centered at the origin.

It is enough to prove that u_p is in $W^{1,(p,\infty)}(B(0,r))$ and in $H^{1,s}(B(0,r))$ for all $r > 0$ and for all $s \in (1,p)$. We can assume without loss of generality that $r > 1$. We fix such s and r.

For every integer $k \ge 1$ we truncate the function u_p on the set $B(0, \frac{1}{k+1})$ and we denote this truncation by $u_{p,k}$. Specifically, for $p = n > 1$ and $k \ge 1$ integer we define $u_{n,k}$ on $\mathbf{R}^n$ by

$$u_{n,k}(x) = \begin{cases} \ln \frac{1}{k+1} & \text{if } 0 \le |x| \le \frac{1}{k+1} \\ u_n(x) = \ln |x| & \text{if } \frac{1}{k+1} \le |x| < \infty. \end{cases}$$

For $p \ne n$ and $k \ge 1$ integer we define $u_{p,k}$ on $\mathbf{R}^n$ by

$$u_{p,k}(x) = \begin{cases} \left(\frac{1}{k+1}\right)^{1-\frac{n}{p}} & \text{if } 0 \le |x| \le \frac{1}{k+1} \\ u_p(x) = |x|^{1-\frac{n}{p}} & \text{if } \frac{1}{k+1} \le |x| < \infty. \end{cases}$$

We notice that $(u_{p,k})_{k\ge1} \subset H^{1,(p,\infty)}(B(0,r))$ is a sequence of Lipschitz functions on $\overline{B}(0,r)$.

Moreover, for every $k \ge 1$ we have $0 \le |u_{p,k}| \le |u_p|$ pointwise in $\mathbf{R}^n$ and $|\nabla u_{p,k}| \le |\nabla u_p|$ almost everywhere in $\mathbf{R}^n$. Thus, the sequence $u_{p,k}$ is bounded in $H^{1,(p,\infty)}(B(0,r))$ and in $H^{1,s}(B(0,r))$ for all $1 < s < p$.

This sequence converges to u_p pointwise in $\mathbf{R}^n \setminus \{0\}$. The aforementioned pointwise convergence on $B^*(0,r)$ together with the reflexivity argument from Heinonen-Kilpeläinen-Martio [19, Theorem 1.32], valid for all integers $n \ge 1$, shows that u_p is in $H^{1,s}(B(0,r))$. Thus, we showed that $u_p \in H^{1,s}(B(0,r))$ for all $1 < s < p$ and all $r > 0$.

We proved that u_p is in $H^{1,s}_{loc}(\mathbf{R}^n)$ for all $s \in (1,p)$. The fact that u_p is in $H^{1,s}_{loc}(\mathbf{R}^n)$ for all $1 < s < p$ coupled with the fact that u_p is in $L^{(p,\infty)}_{loc}(\mathbf{R}^n)$ and $|\nabla u_p|$ is in $L^{(p,\infty)}(\mathbf{R}^n)$ show that u_p is indeed in $W^{1,(p,\infty)}_{loc}(\mathbf{R}^n)$. This finishes the proof.

□

The following theorem shows, among other things, that $H^{1,(p,\infty)}(\Omega) \subsetneq W^{1,(p,\infty)}(\Omega)$; it also shows that the spaces $H^{1,(p,\infty)}_0(\Omega)$, $H^{1,(p,\infty)}(\Omega)$, and $W^{1,(p,\infty)}(\Omega)$ are not reflexive.

Theorem 4.8. *Let $\Omega \subset \mathbf{R}^n$ be an open set, where $n \geq 1$ is an integer and let y be a point in Ω. Suppose $1 < p < \infty$.*

(i) *We have $H^{1,(p,\infty)}(\Omega) \subsetneq W^{1,(p,\infty)}(\Omega) \cap H^{1,(p,\infty)}(\Omega \setminus \{y\})$.*
(ii) *We have $H^{1,(p,q)}(\Omega \setminus \{y\}) \subsetneq H^{1,(p,\infty)}(\Omega \setminus \{y\})$ whenever $1 \leq q < \infty$.*
(iii) *The spaces $H^{1,(p,\infty)}_0(\Omega)$, $H^{1,(p,\infty)}(\Omega)$, and $W^{1,(p,\infty)}(\Omega)$ are not reflexive.*

Proof. The inclusion in (i) follows immediately from Theorem 4.4 and from the definition of the Sobolev-Lorentz spaces $H^{1,(p,q)}$; the inclusion in (ii) follows immediately from Theorem 4.3 (iii) and from the definition of the Sobolev-Lorentz spaces $H^{1,(p,q)}$.

In order to prove the strict inclusions in (i) and (ii) and the non-reflexivity in (iii), we can assume without loss of generality that Ω is a bounded open set in $\mathbf{R}^n$ such that $y = 0 \in \Omega$. Furthermore, we can assume without loss of generality that $\Omega = B(0,r)$ with $r > 1$.

We prove (i) and (ii). We define $u_{r,p} : B(0,r) \to [-\infty, \infty]$ by

$$u_{r,p}(x) = \begin{cases} \ln \frac{|x|}{r} & 0 \leq |x| < r, \text{ if } p = n > 1 \\ |x|^{1-\frac{n}{p}} - r^{1-\frac{n}{p}} & 0 \leq |x| < r, \text{ if } p \neq n. \end{cases}$$

Let $c(n,p,r)$ be a constant that depends on n,p,r, defined by

$$c(n,p,r) = \begin{cases} \ln r & \text{if } p = n > 1 \\ r^{1-\frac{n}{p}} & \text{if } p \neq n. \end{cases} \tag{16}$$

We notice that $u_{r,p} = u_p - c(n,p,r)$ on $B(0,r)$, where u_p is the function from Proposition 4.7, defined on $\mathbf{R}^n$. Thus, from Proposition 4.7 it follows immediately that $u_{r,p}$ is in $W^{1,(p,\infty)}(B(0,r)) \setminus H^{1,(p,\infty)}(B(0,r))$ and that $u_{r,p}$ is not in $H^{1,(p,q)}(B^*(0,r))$ whenever $1 \leq q < \infty$.

Moreover, by mimicking the argument from the proof of Proposition 4.7, we have

$$||\nabla u_{r,p} - \nabla v||_{L^{p,\infty}(B(0,\alpha);\mathbf{R}^n)} \geq ||\nabla u_{r,p}||_{L^{p,\infty}(B(0,r);\mathbf{R}^n)} = |C(n,p)|\,\Omega_n^{1/p}$$

for every $v \in C^\infty(B(0,r))$ and every $\alpha \in (0,r)$, where $C(n,p)$ is the constant from (15).

We notice that $u_{r,p}$ is smooth in $B^*(0,r)$. Since we saw that $u_{r,p}$ is in $W^{1,(p,\infty)}(B(0,r))$, it follows immediately that $u_{r,p} \in H^{1,(p,\infty)}(B^*(0,r))$. This finishes the proof of claims (i) and (ii).

We prove now claim (iii). We modify slightly the reflexivity argument from Proposition 4.7. For every integer $k \geq 1$ we define the functions $u_{r,p,k}$ on $B(0,r)$ by $u_{r,p,k}(x) = u_{p,k}(x) - c(n,p,r), x \in B(0,r)$; here $c(n,p,r)$ is the constant from (16) and $u_{p,k}$ are the functions from Proposition 4.7 (namely the truncations of u_p on $B(0, \frac{1}{k+1})$). Specifically, for $p = n > 1$ and $k \geq 1$ integer we have

$$u_{r,n,k}(x) = \begin{cases} \ln \frac{1}{k+1} - \ln r & \text{if } 0 \leq |x| \leq \frac{1}{k+1} \\ u_{r,n}(x) = \ln \frac{|x|}{r} & \text{if } \frac{1}{k+1} \leq |x| < r. \end{cases}$$

For $p \neq n$ and $k \geq 1$ integer we have

$$u_{r,p,k}(x) = \begin{cases} \left(\frac{1}{k+1}\right)^{1-\frac{n}{p}} - r^{1-\frac{n}{p}} & \text{if } 0 \leq |x| \leq \frac{1}{k+1} \\ u_{r,p}(x) = |x|^{1-\frac{n}{p}} - r^{1-\frac{n}{p}} & \text{if } \frac{1}{k+1} \leq |x| < r. \end{cases}$$

We see that $(u_{r,p,k})_{k\ge 1} \subset H_0^{1,(p,\infty)}(B(0,r))$ is a sequence of Lipschitz functions on $B(0,r)$ that can be extended continuously by 0 on $\partial B(0,r)$. Moreover, for every $k \ge 1$ we have $0 \le |u_{r,p,k}| \le |u_{r,p}|$ pointwise in $B(0,r)$ and $|\nabla u_{r,p,k}| \le |\nabla u_{r,p}|$ almost everywhere in $B(0,r)$.

By using the argument from Proposition 4.7 (ii) with minor modifications, we see that the sequence $u_{r,p,k}$ is bounded in $H_0^{1,(p,\infty)}(B(0,r))$, in $H_0^{1,s}(B(0,r))$ and also in $H^{1,s}(B(0,r))$ for all $1 < s < p$. Since this sequence converges to $u_{r,p}$ pointwise in $B^*(0,r)$ but $u_{r,p}$ is not in $H^{1,(p,\infty)}(B(0,r))$, it follows that $H_0^{1,(p,\infty)}(B(0,r))$ and $H^{1,(p,\infty)}(B(0,r))$ are not reflexive spaces. Moreover, since both these spaces are closed subspaces of $W^{1,(p,\infty)}(B(0,r))$, it follows that the space $W^{1,(p,\infty)}(B(0,r))$ is not reflexive. Thus, we proved claim (ii). This finishes the proof of the theorem.

□

The following lemma shows, among other things, that the product between a function u in $W^{1,(p,q)}(\Omega)$ and a function φ in $C_0^\infty(\Omega)$ yields a function in $H_0^{1,(p,q)}(\Omega)$ if $1 < p < \infty$ and $1 \le q \le \infty$ whenever u and ∇u have absolutely continuous (p,q)-norm.

Lemma 4.9. (See Costea [6, Lemma V.6] and [9, Lemma 3.3.1]). *Let $\Omega \subset \mathbf{R}^n$ be an open set, where $n \ge 1$ is an integer. Suppose $1 < p < \infty$ and $1 \le q \le \infty$. Suppose that $u \in W^{1,(p,q)}(\Omega)$ and that $\varphi \in C_0^\infty(\Omega)$. Then $u\varphi \in W^{1,(p,q)}(\Omega)$ and $\nabla(u\varphi) = u\nabla\varphi + \varphi\nabla u$. Moreover, $u\varphi \in H_0^{1,(p,q)}(\Omega)$ whenever u and ∇u have absolutely continuous (p,q)-norm.*

Proof. We can assume without loss of generality that Ω is bounded. Let $s \in (1,p)$. Then from Theorem 4.3 we have $u \in H^{1,s}(\Omega)$, and hence from Evans [12, p. 247 Theorem 1] it follows that $u\varphi \in H^{1,s}(\Omega)$ and $\nabla(u\varphi) = u\nabla\varphi + \varphi\nabla u$. Since $u\varphi \in L^{(p,q)}(\Omega)$ and $u\nabla\varphi + \varphi\nabla u \in L^{(p,q)}(\Omega;\mathbf{R}^n)$, it follows that $u\varphi \in W^{1,(p,q)}(\Omega)$.

Now suppose that u and ∇u have absolutely continuous (p,q)-norm. (This is always the case when $1 \le q < \infty$). We have to prove that $u\varphi \in H_0^{1,(p,q)}(\Omega)$.

If we multiply u with a function $\widetilde{\eta} \in C_0^\infty(\Omega)$, the first part of the proof shows that both $u\widetilde{\eta}$ and $\nabla(u\widetilde{\eta})$ have absolutely continuous norm whenever u and ∇u have absolutely continuous (p,q)-norm. If the function $\widetilde{\eta}$ is chosen to be 1 on supp φ, then $\widetilde{\eta}\varphi = \varphi$. This allows us to assume without generality that u has compact support in Ω.

Let $\eta \in C_0^\infty(B(0,1))$ be a mollifier. Let $j_0 > 0$ be an integer such that

$$j_0 > (\text{dist}(\text{supp } u, \mathbf{R}^n \setminus \Omega))^{-1}.$$

For $j \ge j_0$ integer we define $u_j : \Omega \to \mathbf{R}$, $u_j(x) = (\eta_j * u)(x)$, where $\eta_j(x) = j^n\eta(jx)$. We notice that $(u_j)_{j\ge j_0} \subset C_0^\infty(\Omega)$. Moreover, since $\eta_j \in C_0^\infty(B(0,j^{-1}))$ are mollifiers and $u \in W^{1,(p,q)}(\Omega)$, it follows via Ziemer [30, Theorem 1.6.1] that $\partial_i u_j = (\partial_i \eta_j) * u = \eta_j * (\partial_i u)$ for all $i = 1,\ldots,n$ and for all integers $j \ge j_0$. Since u and ∇u have absolutely continuous (p,q)-norm, it follows via Theorem 4.5 that u_j converges to u in $H^{1,(p,q)}(\Omega)$ as $j \to \infty$.

This implies, via the first part of the proof that $u_j\varphi, j \ge j_0$ is a sequence in $C_0^\infty(\Omega)$ that converges to $u\varphi$ in $H^{1,(p,q)}(\Omega)$, which means that $u\varphi \in H_0^{1,(p,q)}(\Omega)$. This finishes the proof.

□

Remark 4.10. We notice that the product $u\varphi$ does not necessarily belong to $H_0^{1,(p,\infty)}(\Omega)$ when $u \in W^{1,(p,\infty)}(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$ if ∇u does not have absolutely continuous (p,∞)-norm.

Indeed, let $0 < \alpha < r < \infty$ and let $\Omega = B(0,r)$. Let $u_{r,p}$ be the function from Theorem 4.8. Choose $\varphi_{r,\alpha}$ in $C_0^\infty(\Omega)$ such that $\varphi_{r,\alpha} = 1$ in $B(0,\alpha)$. Then via Lemma 4.9 we have $u_{r,p}\varphi_{r,\alpha} \in W^{1,(p,\infty)}(\Omega)$. It is obvious that $u_{r,p}\varphi_{r,\alpha} = u_{r,p}$ in $B(0,\alpha)$ and hence via Theorem 4.8 it follows that $u_{r,p}\varphi_{r,\alpha}$ does not belong to $H_0^{1,(p,\infty)}(\Omega)$.

Now we prove that if $n \ge 1$ is an integer, $\Omega \subset \mathbf{R}^n$ is an open set and $u \in W^{1,(p,q)}(\Omega)$ is such that u and ∇u have absolutely continuous (p,q)-norm, then $u \in H^{1,(p,q)}(\Omega)$. This result is new for $q = \infty$ and $n \ge 1$. For $1 \le q < \infty$ it yields $H^{1,(p,q)}(\Omega) = W^{1,(p,q)}(\Omega)$, a result proved in Costea [9, Theorem

3.3.4] and Costea [6, Theorem V.9] for $n \geq 2$. Thus, we generalize and improve the result obtained in Costea [9, Theorem 3.3.4] and Costea [6, Theorem V.9].

Theorem 4.11. *Let $\Omega \subset \mathbf{R}^n$ be an open set, where $n \geq 1$ is an integer. Suppose $1 < p < \infty$ and $1 \leq q \leq \infty$. Suppose that $u \in W^{1,(p,q)}(\Omega)$. If u and ∇u have absolutely continuous (p,q)-norm, then $u \in H^{1,(p,q)}(\Omega)$. In particular, $H^{1,(p,q)}(\Omega) = W^{1,(p,q)}(\Omega)$ if $1 \leq q < \infty$.*

Proof. Like in the proof of Ziemer [30, Lemma 2.3.1] and Costea [9, Theorem 3.3.4], we choose open sets $\Omega_0 = \emptyset \subsetneq \Omega_j \subset\subset \Omega_{j+1}, j \geq 1$ such that $\cup_j \Omega_j = \Omega$ and a sequence of functions $\psi_j, j \geq 1$ such that $\psi_j \in C_0^\infty(\Omega_{j+1} \setminus \overline{\Omega}_{j-1})$, $0 \leq \psi_j \leq 1$ for every $j \geq 1$ and $\sum_j \psi_j \equiv 1$ in Ω.

Let $\varepsilon > 0$ be fixed. For every $j \geq 1$, we have via Lemma 4.9 that $u\psi_j$ is in $H_0^{1,(p,q)}(\Omega)$. Moreover, since $\psi_j \in C_0^\infty(\Omega_{j+1} \setminus \overline{\Omega}_{j-1})$, we see that in fact $u\psi_j$ is in $H_0^{1,(p,q)}(\Omega_{j+1} \setminus \overline{\Omega}_{j-1})$ and thus, there exists φ_j in $C_0^\infty(\Omega_{j+1} \setminus \overline{\Omega}_{j-1})$ such that

$$||\varphi_j - u\psi_j||_{1,(p,q);\Omega} \leq ||\varphi_j - u\psi_j||_{L^{(p,q)}(\Omega)} + ||\nabla\varphi_j - \nabla(u\psi_j)||_{L^{(p,q)}(\Omega;\mathbf{R}^n)} < \frac{\varepsilon}{2^j}$$

for all $j \geq 1$. If we define $\varphi \equiv \sum_{j \geq 1} \varphi_j$, we see that $\varphi \in C^\infty(\Omega)$ because φ can be written as a finite sum of the functions $\varphi_i \in C_0^\infty(\Omega)$ on every bounded open set $U \subset\subset \Omega$. Moreover,

$$||\varphi - u||_{1,(p,q);\Omega} = ||\sum_{j\geq 1}(\varphi_j - u\psi_j)||_{1,(p,q);\Omega} \leq \sum_{j \geq 1} ||\varphi_j - u\psi_j||_{1,(p,q);\Omega} < \sum_{j\geq 1} \frac{\varepsilon}{2^j} = \varepsilon.$$

This finishes the proof of the theorem.

□

Now we prove that if $n \geq 1$ is an integer and $u \in W^{1,(p,q)}(\mathbf{R}^n)$ is such that u and ∇u have absolutely continuous (p,q)-norm, then $u \in H_0^{1,(p,q)}(\mathbf{R}^n)$. This result is new for $q = \infty$ and $n \geq 1$. For $1 \leq q < \infty$ it yields $H^{1,(p,q)}(\mathbf{R}^n) = H_0^{1,(p,q)}(\mathbf{R}^n)$, a result proved in Costea [9, Theorem 3.3.6] and Costea [6, Theorem V.16] for $n \geq 2$. Thus, we generalize and improve the result obtained in Costea [9, Theorem 3.3.6] and Costea [6, Theorem V.16].

Theorem 4.12. *Suppose $1 < p < \infty$ and $1 \leq q \leq \infty$. Suppose that $u \in W^{1,(p,q)}(\mathbf{R}^n)$, where $n \geq 1$ is an integer. If u and ∇u have absolutely continuous (p,q)-norm, then $u \in H_0^{1,(p,q)}(\mathbf{R}^n)$. In particular, $H^{1,(p,q)}(\mathbf{R}^n) = H_0^{1,(p,q)}(\mathbf{R}^n)$ if $1 \leq q < \infty$.*

Proof. Let $u \in W^{1,(p,q)}(\mathbf{R}^n)$ such that u and ∇u have absolutely continuous (p,q)-norm. (This is always the case when $1 \leq q < \infty$). Then from Theorem 4.11 it follows that u is in fact in $H^{1,(p,q)}(\mathbf{R}^n)$.

For $j = 1, 2, \ldots$ choose functions $\varphi_j \in C_0^\infty(B(0, j+1))$, $0 \leq \varphi_j \leq 1$, such that $\varphi_j(x) = 1$ for each $x \in \overline{B}(0, j)$. Moreover, we choose these functions φ_j to be radial and 2-Lipschitz for all $j \geq 1$. Then $u_j := u\varphi_j \in H_0^{1,(p,q)}(B(0, j+1))$ for all $j \geq 1$ via Lemma 4.9.

Fix $\varepsilon > 0$. For every $j \geq 1$ choose $\psi_j \in C_0^\infty(B(0, j+1))$ such that

$$||\psi_j - u_j||_{1,(p,q);\mathbf{R}^n} = ||\psi_j - u\varphi_j||_{1,(p,q);\mathbf{R}^n} < \frac{\varepsilon}{4}.$$

For every $j \geq 1$ integer we have, via the definition of the $H^{1,(p,q)}$-norm and via Lemma 4.9

$$\begin{aligned} ||u - u_j||_{1,(p,q);\mathbf{R}^n} &\leq ||u - u_j||_{L^{(p,q)}(\mathbf{R}^n)} + ||\nabla u - \nabla u_j||_{L^{(p,q)}(\mathbf{R}^n;\mathbf{R}^n)} \\ &\leq ||u(1-\varphi_j)||_{L^{(p,q)}(\mathbf{R}^n)} + ||(1-\varphi_j)\nabla u||_{L^{(p,q)}(\mathbf{R}^n;\mathbf{R}^n)} \\ &\quad + ||u\nabla\varphi_j||_{L^{(p,q)}(\mathbf{R}^n;\mathbf{R}^n)}. \end{aligned}$$

Since $0 \leq \varphi_j \leq 1$ is a 2-Lipschitz smooth function supported in $B(0, j+1)$ such that $\varphi_j = 1$ in $\overline{B}(0, j)$, this yields

$$\begin{aligned} ||u - u_j||_{1,(p,q);\mathbf{R}^n} &\leq ||u\chi_{\mathbf{R}^n \setminus B(0,j)}||_{L^{(p,q)}(\mathbf{R}^n)} \\ &\quad + ||\nabla u \chi_{\mathbf{R}^n \setminus B(0,j)}||_{L^{(p,q)}(\mathbf{R}^n;\mathbf{R}^n)} \\ &\quad + ||\nabla \varphi_j||_{L^\infty(\mathbf{R}^n)} ||u\chi_{\mathbf{R}^n \setminus B(0,j)}||_{L^{(p,q)}(\mathbf{R}^n;\mathbf{R}^n)} \\ &\leq 3||u\chi_{\mathbf{R}^n \setminus B(0,j)}||_{L^{(p,q)}(\mathbf{R}^n)} \\ &\quad + ||\nabla u \chi_{\mathbf{R}^n \setminus B(0,j)}||_{L^{(p,q)}(\mathbf{R}^n;\mathbf{R}^n)} \end{aligned}$$

for all $j \geq 1$.

Since u and ∇u have absolutely continuous (p, q)-norm, we can choose an integer $j_0 > 1$ such that

$$||u\chi_{\mathbf{R}^n \setminus B(0,j)}||_{L^{(p,q)}(\mathbf{R}^n)} + ||\nabla u \chi_{\mathbf{R}^n \setminus B(0,j)}||_{L^{(p,q)}(\mathbf{R}^n;\mathbf{R}^n)} < \frac{\varepsilon}{4}$$

for all $j \geq j_0$.

Thus, $(\psi_j)_{j \geq 1} \subset C_0^\infty(\mathbf{R}^n)$ and

$$\begin{aligned} ||\psi_j - u||_{1,(p,q);\mathbf{R}^n} &\leq ||\psi_j - u_j||_{1,(p,q);\mathbf{R}^n} + ||u - u_j||_{1,(p,q);\mathbf{R}^n} \\ &< \frac{\varepsilon}{4} + 3||u\chi_{\mathbf{R}^n \setminus B(0,j)}||_{L^{(p,q)}(\mathbf{R}^n)} \\ &\quad + ||\nabla u \chi_{\mathbf{R}^n \setminus B(0,j)}||_{L^{(p,q)}(\mathbf{R}^n;\mathbf{R}^n)} < \varepsilon \end{aligned}$$

for all $j \geq j_0$. This finishes the proof of the theorem.

□

We prove now that $W^{1,(p,q_1)}(\Omega) \subsetneq W^{1,(p,q_2)}(\Omega)$ whenever $1 < p < \infty$ and $1 \leq q_1 < q_2 \leq \infty$.

Theorem 4.13. *Let $n \geq 1$ be an integer and $r > 0$ be a positive number. Suppose $1 < p < \infty$ and $1 \leq q_1 < q_2 \leq \infty$. Let α be a number in $(0, 1]$ such that $1 \leq q_1 \leq \frac{1}{\alpha} < q_2 \leq \infty$. Let $v_{r,\alpha,n,p} : B(0, r) \to [0, \infty]$ be the function defined in (7).*

Then

(i) $v_{r,\alpha,n,p} \in H_0^{1,(p,q_2)}(B(0, r)) \setminus H^{1,(p,q_1)}(B(0, r))$.

(ii) $v_{r,\alpha,n,p} \in H^{1,(p,q_2)}(B^*(0, r)) \setminus H^{1,(p,q_1)}(B^*(0, r))$.

Proof. By choosing q_3 such that $\frac{1}{\alpha} < q_3 < q_2$ if necessary, we can assume without loss of generality via Theorem 4.3 (iii) that $q_2 < \infty$ throughout the proof of this theorem.

Since n, p, r, α, q_1 and q_2 are fixed here, we simplify the notations throughout the proof of the theorem. We let $v_{n,p} := v_{r,\alpha,n,p}$ and $f_{\alpha,p} := f_{rad,r,\alpha,p}$, where $f_{rad,r,\alpha,p}$ is the function defined in (6).

Since

$$||\nabla v_{n,p}||_{L^{p,q_1}(B^*(0,r);\mathbf{R}^n)} = \infty,$$

it follows immediately via Theorem 4.11 that $v_{n,p} \notin H^{1,(p,q_1)}(B^*(0, r))$ and consequently $v_{n,p} \notin H^{1,(p,q_1)}(B(0, r)) = W^{1,(p,q_1)}(B(0, r))$.

We want to show that $v_{n,p} \in H_0^{1,(p,q_2)}(B(0, r))$. In order to do that, we resort to a truncation argument and we invoke Theorem 4.1.

We know from the proof of Theorem 3.5 that $f_{\alpha,p}$ is in $C^\infty((0, r))$, positive and strictly decreasing on $(0, r)$. Moreover, we have $\lim_{t \to 0} f_{\alpha,p}(t) = \infty$ when $1 < p \leq n$ and $\lim_{t \to 0} f_{\alpha,p}(t) < \infty$ when $n < p < \infty$.

For every integer $k \geq 1$ we truncate the function $v_{n,p}$ on the set $B(0, \frac{r}{k+1})$ and we denote this truncation by $v_{n,p,k}$. Specifically, for $k \geq 1$ integer we define $u_{n,p,k}$ on $B(0, r)$ by

$$v_{n,p,k}(x) = \begin{cases} f_{\alpha,p}(\frac{r}{k+1}) & \text{if } 0 \leq |x| \leq \frac{r}{k+1} \\ v_{n,p}(x) = f_{\alpha,p}(|x|) & \text{if } \frac{r}{k+1} < |x| < r. \end{cases}$$

It is easy to see that $0 \le v_{n,p,k} \le v_{n,p}$ pointwise in $B(0,r)$ for all $k \ge 1$. Moreover, all the functions $v_{n,p,k}$ are Lipschitz on $B(0,r)$ and can be extended continuously by 0 on $\partial B(0,r)$. More precisely, for all $k \ge 1$ we have

$$\nabla v_{n,p,k}(x) = \begin{cases} 0 & \text{if } 0 \le |x| < \frac{r}{k+1} \\ \nabla v_{n,p}(x) = f'_{\alpha,p}(|x|)\frac{x}{|x|} & \text{if } \frac{r}{k+1} < |x| < r. \end{cases}$$

In particular, for every $k \ge 1$ we have $|\nabla v_{n,p,k}| \le |\nabla v_{n,p}|$ almost everywhere in $B(0,r)$. Thus, we have that $(v_{n,p,k})_{k \ge 1} \subset H_0^{1,(p,q_2)}(B(0,r))$. We claim that the sequence $v_{n,p,k}$ is bounded in $H_0^{1,(p,q_2)}(B(0,r))$ and in $H^{1,(p,q_2)}(B^*(0,r))$.

We study the cases $n=1$ and $n>1$ separately.

Case I. We suppose first that $n=1$. Then $p>n$ and from Theorem 3.5 (iv) it follows that $v_{n,p}$ is continuous and bounded on $B(0,r)$. The boundedness of the sequence $v_{n,p,k}$ in $H_0^{1,(p,q_2)}(B(0,r))$ and in $H^{1,(p,q_2)}(B^*(0,r))$ is immediate in this case since $0 \le v_{n,p,k} \le v_{n,p}$ pointwise in $B(0,r)$, $|\nabla v_{n,p}| \in L^{(p,q_2)}(B(0,r))$ and since $|\nabla v_{n,p,k}| \le |\nabla v_{n,p}|$ almost everywhere in $B(0,r)$ for every $k \ge 1$.

Case II. We assume now that $n>1$. Via Theorem 4.2 (ii) we have

$$\begin{aligned} ||v_{n,p,k}||_{L^{p,q_2}(B(0,r))} &\le C(n,p,q_2)\,|\Omega|^{\frac{1}{n}}\,||\nabla v_{n,p,k}||_{L^{p,q_2}(B(0,r);\mathbf{R}^n)} \\ &\le C(n,p,q_2)\,|\Omega|^{\frac{1}{n}}\,||\nabla v_{n,p}||_{L^{p,q_2}(B(0,r);\mathbf{R}^n)} \end{aligned}$$

for every $k \ge 1$ integer.

Thus, we proved that the sequence $v_{n,p,k}$ is bounded in $H_0^{1,(p,q_2)}(B(0,r))$ and in $H^{1,(p,q_2)}(B^*(0,r))$ whenever $n \ge 1$, $1<p<\infty$, and $1<q_2<\infty$. The reflexivity of these two spaces and the pointwise convergence of $v_{n,p,k}$ to $v_{n,p}$ on $B^*(0,r)$ imply immediately via Theorem 4.1 that $v_{n,p}$ is in fact in $H_0^{1,(p,q_2)}(B(0,r))$ and in $H^{1,(p,q_2)}(B^*(0,r))$. Moreover, by invoking Theorem 5.5 (i) for $n=1$ and respectively Theorem 5.6 (iv) for $n>1$, we see that $v_{n,p}$ is in fact Hölder continuous in $\overline{B}(0,r)$ with exponent $1-\frac{n}{p}$. This finishes the proof.

□

4.4. **Chain Rule Results.** We recall the chain rule property for the Sobolev-Lorentz spaces, proved in Costea [9] for $n \ge 2$.

Theorem 4.14. (See Costea [9, Theorem 3.4.1]). *Let $\Omega \subset \mathbf{R}^n$ be an open set, where $n \ge 1$ is an integer. Suppose $1<p<\infty$ and $1 \le q \le \infty$. Suppose that $f \in C^1(\mathbf{R})$, $f(0)=0$ and f' is bounded. If $u \in W^{1,(p,q)}(\Omega)$, then $f \circ u \in W^{1,(p,q)}(\Omega)$ and*

$$\nabla(f \circ u) = f'(u)\nabla u.$$

Moreover, if $u \in H_0^{1,(p,q)}(\Omega)$, then $f \circ u \in H_0^{1,(p,q)}(\Omega)$.

Proof. We have

$$|f \circ u(x)| = |f(u(x)) - f(0)| \le ||f'||_{L^\infty(\mathbf{R})}|u(x)| \text{ for a.e. } x \text{ in } \Omega, \tag{17}$$

which implies that $f \circ u \in L^{(p,q)}(\Omega)$.

Let $s \in (1,p)$ be fixed. We have that $u \in W^{1,(p,q)}(\Omega)$, hence by Theorem 4.3 it follows that $u \in H^{1,s}_{loc}(\Omega)$. This and (17) imply via Ziemer [30, Theorem 2.1.11] that $f \circ u \in H^{1,s}_{loc}(\Omega)$ and

$$\nabla(f \circ u) = f'(u)\nabla u.$$

Thus, we have $f \circ u \in L^{(p,q)}(\Omega)$, $\nabla(f \circ u) = f'(u)\nabla u \in L^{(p,q)}(\Omega;\mathbf{R}^n)$, which implies that $f \circ u \in W^{1,(p,q)}(\Omega)$.

We want to prove that $f \circ u \in H_0^{1,(p,q)}(\Omega)$ if $u \in H_0^{1,(p,q)}(\Omega)$. This was done in Costea [9, Theorem 3.4.1] in the case $n \ge 2$, but the proof is valid for $n=1$ as well. We present it for the convenience of the reader.

Suppose that $u \in H_0^{1,(p,q)}(\Omega)$. Then it follows immediately that u and ∇u have absolutely continuous (p,q)-norm. From the first part of the proof we already know that $f \circ u \in W^{1,(p,q)}(\Omega)$ because $u \in H_0^{1,(p,q)}(\Omega) \subset W^{1,(p,q)}(\Omega)$. Let $u_j, j \geq 1$ be a sequence of functions in $C_0^\infty(\Omega)$ that converges to u in $H_0^{1,(p,q)}(\Omega)$. Without loss of generality, we can assume that $u_j \to u$ pointwise almost everywhere in Ω. Since the functions u_j are compactly supported in Ω and $f(0) = 0$, it follows that the functions $f \circ u_j$ are compactly supported in Ω. Moreover, since the functions u_j are in $C^1(\Omega)$ and f is in $C^1(\mathbf{R})$, it follows that the functions $f \circ u_j$ are in $C^1(\Omega)$. Thus, $f \circ u_j, j \geq 1$ is a sequence of functions in $C_0^1(\Omega) \subset H_0^{1,(p,q)}(\Omega)$ with $\nabla(f \circ u_j) = f'(u_j)\nabla u_j, j \geq 1$. Since f' is bounded on $\mathbf{R}$, we have

$$|(f \circ u_j)(x) - (f \circ u)(x)| \leq ||f'||_{L^\infty(\mathbf{R})}|u_j(x) - u(x)| \text{ for a.e. } x \text{ in } \Omega.$$

This implies that $f \circ u_j$ converges to $f \circ u$ in $L^{(p,q)}(\Omega)$.

We have

$$\begin{aligned} ||f'(u_j)\nabla u_j - f'(u)\nabla u||_{L^{(p,q)}(\Omega;\mathbf{R}^n)} &\leq ||f'||_{L^\infty(\mathbf{R})}||\nabla u_j - \nabla u||_{L^{(p,q)}(\Omega;\mathbf{R}^n)} \\ &\quad + ||(f'(u_j) - f'(u))\nabla u||_{L^{(p,q)}(\Omega;\mathbf{R}^n)}. \end{aligned}$$

The first term of the right-hand side trivially converges to 0. The second term of the right-hand side converges to 0 via Bennett-Sharpley [1, Proposition I.3.6] since ∇u has absolutely continuous (p,q)-norm, f' is bounded and $f'(u_j)$ converges to $f'(u)$ pointwise almost everywhere in Ω. Consequently, the sequences $f(u_j), f'(u_j)\nabla u_j$ converge to $f(u), f'(u)\nabla u$ respectively and thus $\nabla(f \circ u) = f'(u)\nabla u$. This finishes the proof.

□

Recall the notation

$$u^+ = \max(u,0) \text{ and } u^- = \min(u,0).$$

Lemma 4.15. (See Costea [6, Lemma V.12] and [9, Lemma 3.4.4]). *Suppose* $1 < p < \infty$ *and* $1 \leq q \leq \infty$. *If* $u \in W^{1,(p,q)}(\Omega)$, *then* $u^+ \in W^{1,(p,q)}(\Omega)$ *and*

$$\nabla u^+ = \begin{cases} \nabla u & \text{if } u > 0 \\ 0 & \text{if } u \leq 0. \end{cases}$$

Proof. Let $s \in (1,p)$ be fixed. From Theorem 4.3 we have that $u \in H^{1,s}_{loc}(\Omega)$. Via Evans [12, p. 291-292, Exercise 20] it follows that

$$\nabla u^+ = \begin{cases} \nabla u & \text{if } u > 0 \\ 0 & \text{if } u \leq 0. \end{cases}$$

But in that case $\nabla u^+ \in L^{(p,q)}(\Omega;\mathbf{R}^n)$ since ∇u is in $L^{(p,q)}(\Omega;\mathbf{R}^n)$ and $|\nabla u^+(x)| \leq |\nabla u(x)|$ for almost every x in Ω. We also have $u^+ \in L^{(p,q)}(\Omega)$ since $|u^+| \leq |u|$ in Ω and $u \in L^{(p,q)}(\Omega)$. So we have in fact that $u^+ \in W^{1,(p,q)}(\Omega)$. The claim is proved.

□

From Theorem 4.11 and Lemma 4.15 it follows immediately that the space $H^{1,(p,q)}(\Omega)$ is closed under truncations from above by nonnegative numbers and from below by negative numbers whenever $1 < p < \infty$ and $1 \leq q < \infty$. Moreover, we have the following density result.

Theorem 4.16. (See Costea [9, Theorem 3.4.5]). *Suppose* $1 < p < \infty$ *and* $1 \leq q < \infty$. *Bounded functions in* $H^{1,(p,q)}(\Omega)$ *are dense in* $H^{1,(p,q)}(\Omega)$.

It is important to notice that the Sobolev-Lorentz space $W^{1,(p,q)}(\Omega)$ is a lattice.

Theorem 4.17. (See Costea [6, Theorem V.13] and [9, Theorem 3.4.6]). *Suppose* $1 < p < \infty$ *and* $1 \leq q \leq \infty$. *If* u *and* v *are in* $W^{1,(p,q)}(\Omega)$, *then* $\max(u,v)$ *and* $\min(u,v)$ *are in* $W^{1,(p,q)}(\Omega)$ *with*

$$\nabla \max(u,v)(x) = \begin{cases} \nabla u(x) & \text{if } u(x) \geq v(x) \\ \nabla v(x) & \text{if } v(x) \geq u(x) \end{cases}$$

and

$$\nabla \min(u,v)(x) = \begin{cases} \nabla u(x) & \text{if } u(x) \leq v(x) \\ \nabla v(x) & \text{if } v(x) \leq u(x). \end{cases}$$

In particular, $|u| = u^+ - u^-$ *belongs to* $W^{1,(p,q)}(\Omega)$.

Lemma 4.18. (See Costea [6, Lemma V.14] and [9, Lemma 3.4.7]). *Suppose* $1 < p < \infty$ *and* $1 \leq q < \infty$. *If* $u_j, v_j \in H^{1,(p,q)}(\Omega)$ *are such that* $u_j \to u$ *and* $v_j \to v$ *in* $H^{1,(p,q)}(\Omega)$, *then* $\min(u_j, v_j) \to \min(u,v)$ *and similarly* $\max(u_j, v_j) \to \max(u,v)$ *in* $H^{1,(p,q)}(\Omega)$.

We recall next that the space $H_0^{1,(p,q)}(\Omega)$ is also a lattice whenever $1 < p < \infty$ and $1 \leq q \leq \infty$.

Theorem 4.19. (See Costea [6, Theorem V.15] and [9, Theorem 3.4.8]). *Suppose* $1 < p < \infty$ *and* $1 \leq q \leq \infty$. *If* u *and* v *are in* $H_0^{1,(p,q)}(\Omega)$, *then* $\max(u,v)$ *and* $\min(u,v)$ *are in* $H_0^{1,(p,q)}(\Omega)$. *Moreover, if* $u \in H_0^{1,(p,q)}(\Omega)$ *is nonnegative, then there exists a sequence of nonnegative functions* $\varphi_j \in C_0^\infty(\Omega)$ *that converges to* u *in* $H_0^{1,(p,q)}(\Omega)$.

We have a result analogous to Theorem 4.16 for $H_0^{1,(p,q)}(\Omega)$ whenever $1 < p < \infty$ and $1 \leq q \leq \infty$.

Theorem 4.20. (See Costea [9, Theorem 3.4.9]). *Suppose* $1 < p < \infty$ *and* $1 \leq q \leq \infty$. *Bounded functions in* $H_0^{1,(p,q)}(\Omega)$ *are dense in* $H_0^{1,(p,q)}(\Omega)$.

It is easy to see that if a function u is in $H_0^{1,(p,q)}(\Omega)$, then u and its distributional gradient ∇u must have absolutely continuous (p,q)-norm. Next we give a sufficient condition for membership in $H_0^{1,(p,q)}(\Omega)$.

Lemma 4.21. *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded open set, where* $n \geq 1$ *is an integer. Suppose* $1 < p < \infty$ *and* $1 \leq q \leq \infty$. *Suppose that* u *is a function in* $W^{1,(p,q)}(\Omega)$ *such that* $\lim_{x\to y} u(x) = 0$ *for all* $y \in \partial\Omega$. *If* ∇u *has absolutely continuous* (p,q)*-norm, then* $u \in H_0^{1,(p,q)}(\Omega)$.

Proof. We first show that u has absolutely continuous (p,q)-norm if u satisfies the hypotheses of this lemma, a fact which is trivial when $1 \leq q < \infty$. We have to consider the cases $n = 1$ and $n \geq 2$ separately.

Case I. We assume first that $n = 1$. Then via Theorem 5.5 it follows that u has a version $\overline{u}$ that is Hölder continuous in $\overline{\Omega}$ with exponent $1 - \frac{1}{p}$. Without loss of generality we can assume that u is Hölder continuous in $\overline{\Omega}$ with exponent $1 - \frac{1}{p}$. Thus, it follows that u has absolutely continuous (p,q)-norm on Ω if $n = 1$ because u is continuous on the bounded set $\overline{\Omega}$.

Case II. We assume now that $n \geq 2$. Let s be chosen in $(1,p)$. Since Ω is bounded, it follows via Theorem 4.3 that u is in $H^{1,s}(\Omega)$. Thus, it follows via Heinonen-Kilpeläinen-Martio [19, Lemma 1.26] that $u \in H_0^{1,s}(\Omega)$. If in addition, s is chosen such that $p < \frac{ns}{n-s}$, then we have via Sobolev's embedding theorem for $H_0^{1,s}(\Omega)$ (see Gilbarg-Trudinger [15, Theorem 7.10]) that in fact $u \in L^{\frac{ns}{n-s}}(\Omega)$. This, (1), Theorem 3.7 and Bennett-Sharpley [1, Proposition IV.4.2 and Lemma IV.4.5] show that actually u has absolutely continuous (p,q)-norm for $n \geq 2$ if it satisfies the hypotheses of the lemma.

Since we now know that u and ∇u have absolutely continuous (p,q)-norm whenever u satisfies the hypotheses of the lemma, it follows via Theorem 4.11 that u is in fact in $H^{1,(p,q)}(\Omega)$.

By recalling that $u = u^+ + u^-$, it follows immediately via Lemma 4.15 that both ∇u^+ and ∇u^- have absolutely continuous p,q-norm since ∇u has absolutely continuous p,q-norm and since $|\nabla u^+|, |\nabla u^-| \leq |\nabla u|$ a.e. in Ω.

We also notice that both u^+ and u^- have absolutely continuous (p,q)-norm since $|u^+|, |u^-| \leq |u|$ and since u has absolutely continuous (p,q)-norm. Moreover, $\lim_{x\to y} u^+(x) = \lim_{x\to y} u^-(x) = 0$ for all $y \in \partial\Omega$ since $\lim_{x \to y} u(x) = 0$ for all $y \in \partial\Omega$. Hence, u^+ and u^- satisfy the hypotheses of the lemma if u does, which implies via Theorem 4.11 that u^+ and u^- are in fact in $H^{1,(p,q)}(\Omega)$. Thus, it is enough to prove the claim of the lemma for u^+ and u^-. This means that we can assume without loss of generality that $u \geq 0$.

Fix $\varepsilon > 0$. Let $u_\varepsilon = (u-\varepsilon)^+ = \max(u-\varepsilon, 0)$. Then u_ε has compact support in Ω. Moreover, via Theorem 4.17, we see that $u_\varepsilon \in W^{1,(p,q)}(\Omega)$ and

$$\nabla u_\varepsilon = \begin{cases} \nabla u & \text{if } u > \varepsilon \\ 0 & \text{if } 0 \leq u \leq \varepsilon. \end{cases}$$

We now show that u_ε is in $H_0^{1,(p,q)}(\Omega)$. The function u_ε has absolutely continuous (p,q)-norm since $0 \leq u_\varepsilon \leq u$ pointwise in Ω and since u has absolutely continuous (p,q)-norm. Similarly, ∇u_ε has absolutely continuous (p,q)-norm since $0 \leq |\nabla u_\varepsilon| \leq |\nabla u|$ almost everywhere in Ω and since ∇u has absolutely continuous (p,q)-norm. These two facts plus the membership of u_ε in $W^{1,(p,q)}(\Omega)$ yield the membership of u_ε in $H_0^{1,(p,q)}(\Omega)$ via Lemma 4.9.

We now show that u_ε converges to u in $W^{1,(p,q)}(\Omega)$. Indeed, we see that $0 \leq u - u_\varepsilon \leq \varepsilon$ pointwise on the bounded set Ω, which implies

$$\lim_{\varepsilon \to 0} ||u_\varepsilon - u||_{L^{(p,q)}(\Omega)} = 0.$$

We also see via Theorem 4.17 and the definition of u_ε that

$$\nabla u - \nabla u_\varepsilon = \nabla u \chi_{0 < u \leq \varepsilon} \text{ a.e. in } \Omega.$$

This and the absolute continuity of the (p,q)-norm of ∇u yield

$$\lim_{\varepsilon \to 0} ||\nabla u_\varepsilon - \nabla u||_{L^{(p,q)}(\Omega;\mathbf{R}^n)} = \lim_{\varepsilon \to 0} ||\nabla u \chi_{0 < u \leq \varepsilon}||_{L^{(p,q)}(\Omega;\mathbf{R}^n)} = 0.$$

This finishes the proof.

□

5. Hölder continuity of functions in Sobolev-Lorentz spaces

In this section we extend some of the known classical embedding theorems to the spaces $H_0^{1,(p,q)}(\Omega)$, $C_0(\Omega) \cap W^{1,(p,q)}(\Omega)$ and $W_{loc}^{1,(p,q)}(\Omega)$ for $1 \leq n < p < \infty$ and $1 \leq q \leq \infty$, where $\Omega \subset \mathbf{R}^n$ is open. First we recall the definition of Hölder continuous functions with exponent $0 < \alpha < 1$.

Definition 5.1. (See Gilbarg-Trudinger [15, p. 52-53] and Ziemer [30, p. 2-3]). Let $n \geq 1$ and $0 < \alpha < 1$. Let u be a function defined on a set $D \subset \mathbf{R}^n$. We say that u is *Hölder continuous in D with exponent α* if the quantity

$$[u]_{0,\alpha;D} := \sup_{x,y \in D, x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha}$$

is finite. We say that u is *locally Hölder continuous in D with exponent α* if u is Hölder continuous with exponent α on compact subsets of D.

Let $\Omega \subset \mathbf{R}^n$ be an open set. Let u be a continuous function on $\overline{\Omega}$. We say that u is in $C^{0,\alpha}(\overline{\Omega})$ if u is Hölder continuous in Ω with exponent α.

Before we state and prove these embedding results, we need to prove an extension result for functions in $H_0^{1,(p,q)}(\Omega)$.

Proposition 5.2. *Let $\Omega \subset \widetilde{\Omega}$ be two open sets in $\mathbf{R}^n$, where $n \geq 1$ is an integer. Suppose $1 < p < \infty$ and $1 \leq q \leq \infty$. Let u be a function in $H_0^{1,(p,q)}(\Omega)$ and let $\widetilde{u}$ be the extension of u by zero to $\widetilde{\Omega}$. Then $\widetilde{u} \in H_0^{1,(p,q)}(\widetilde{\Omega})$.*

Proof. Let $u_k, k \geq 1$ be a sequence of functions in $C_0^\infty(\Omega)$ such that u_k converges to u in $H_0^{1,(p,q)}(\Omega)$. By passing to a subsequence if necessary, we can assume without loss of generality that u_k converges to u pointwise almost everywhere in Ω and that ∇u_k converges to ∇u pointwise almost everywhere in Ω. For every $k \geq 1$ let $\widetilde{u}_k$ be the extension of u_k by 0 to $\widetilde{\Omega}$. Then $\widetilde{u}_k \in C_0^\infty(\widetilde{\Omega})$ for all $k \geq 1$ and

$$||\widetilde{u}_k - \widetilde{u}_l||_{H_0^{1,(p,q)}(\widetilde{\Omega})} = ||u_k - u_l||_{H_0^{1,(p,q)}(\Omega)}$$

for all $l, k \geq 1$. Hence the sequence $(\widetilde{u}_k)_{k\geq 1}$ is fundamental in $H_0^{1,(p,q)}(\widetilde{\Omega})$ since the sequence $(u_k)_{k\geq 1}$ is fundamental in $H_0^{1,(p,q)}(\Omega)$. Thus, the sequence $(\widetilde{u}_k)_{k\geq 1}$ converges in $H_0^{1,(p,q)}(\widetilde{\Omega})$ to some function $v \in H_0^{1,(p,q)}(\widetilde{\Omega})$. Since $\widetilde{u}_k$ converges to $\widetilde{u}$ in $L^{p,q}(\widetilde{\Omega})$ and pointwise almost everywhere in $\widetilde{\Omega}$, it follows in fact that $\widetilde{u} = v$ almost everywhere in $\widetilde{\Omega}$. Thus, $\widetilde{u} \in H_0^{1,(p,q)}(\widetilde{\Omega})$. This finishes the proof. □

We prove later that if $n = 1$ and $\Omega \subset \mathbf{R}$ is an open interval, then all the functions in $H_0^{1,(p,q)}(\Omega)$ and in $W^{1,(p,q)}(\Omega)$ have representatives that are Hölder continuous in $\overline{\Omega}$ with exponent $1 - \frac{1}{p}$. The following result is the first step in this direction.

Proposition 5.3. *Suppose $n = 1 < p < \infty$ and $1 \leq q \leq \infty$. Let $\Omega \subset \mathbf{R}$ be an open interval. If $u \in C^1(\Omega)$ and $x, y \in \Omega$ with $x < y$, then*

$$|u(x) - u(y)| \leq C(p,q)|x-y|^{1-\frac{1}{p}}||u'||_{L^{p,q}((x,y))}, \tag{18}$$

where

$$C(p,q) = \begin{cases} 1 & \text{if } q = 1 \\ \left(\frac{p'}{q'}\right)^{\frac{1}{q'}} & \text{if } 1 < q \leq \infty. \end{cases} \tag{19}$$

Proof. Let $x, y \in \Omega$ with $x < y$ and $u \in C^1(\Omega)$. Then $u(y) - u(x) = \int_x^y u'(t)dt$. By taking absolute values on both sides and using Theorem 3.7, we obtain

$$\begin{aligned} |u(x) - u(y)| &\leq \int_x^y |u'(t)|dt = ||u'||_{L^1((x,y))} \leq ||u'||_{L^{p,q}((x,y))}||1||_{L^{p',q'}((x,y))} \\ &= |x-y|^{1-\frac{1}{p}}||1||_{L^{p',q'}((0,1))}||u'||_{L^{p,q}((x,y))}. \end{aligned}$$

This finishes the proof of the theorem, since $||1||_{L^{p',q'}((0,1))} = C(p,q)$, the constant defined in (19). □

Definition 5.4. We say that the function $\overline{u}$ defined on Ω is a version of a given function u on Ω if $u = \overline{u}$ a.e. in Ω.

Now we prove (among other things) that if $n = 1$ and $\Omega \subset \mathbf{R}$ is an open interval, then all the functions in $H_0^{1,(p,q)}(\Omega)$ and in $W^{1,(p,q)}(\Omega)$ have representatives that are Hölder continuous in $\overline{\Omega}$ with exponent $1 - \frac{1}{p}$.

Theorem 5.5. *Suppose $n = 1 < p < \infty$ and $1 \leq q \leq \infty$. Let $\Omega \subset \mathbf{R}$ be an open set. Let $C(p,q)$ be the constant from (19).*

(i) *Suppose that Ω is an interval. If $u \in W^{1,(p,q)}(\Omega)$, then there exists a version $\overline{u} \in C(\overline{\Omega})$ that is Hölder continuous in $\overline{\Omega}$ with exponent $1 - \frac{1}{p}$ and*

$$[\overline{u}]_{0,1-\frac{1}{p};\overline{\Omega}} \leq C(p,q)||u'||_{L^{p,q}(\Omega)}.$$

(ii) *If $u \in W_{loc}^{1,(p,q)}(\Omega)$, then there exists a version $\overline{u} \in C(\Omega)$ that is locally Hölder continuous in Ω with exponent $1 - \frac{1}{p}$ and*

$$[\overline{u}]_{0,1-\frac{1}{p};\overline{\Omega'}} \leq C(p,q)||u'||_{L^{p,q}(\Omega')},$$

whenever Ω' is an open subinterval of Ω such that $\Omega' \subset\subset \Omega$. Moreover, if $u' \in L^{(p,q)}(\Omega)$ and Ω is an interval, then $\overline{u}$ is Hölder continuous in $\overline{\Omega}$ with exponent $1-\frac{1}{p}$ and

$$[\overline{u}]_{0,1-\frac{1}{p};\overline{\Omega}} \leq C(p,q)||u'||_{L^{p,q}(\Omega)}.$$

(iii) *If $u \in H_0^{1,(p,q)}(\Omega)$, then there exists a version $\overline{u} \in C(\overline{\Omega})$ that is Hölder continuous in $\overline{\Omega}$ with exponent $1-\frac{1}{p}$ and*

$$[\overline{u}]_{0,1-\frac{1}{p};\overline{\Omega}} \leq C(p,q)||u'||_{L^{p,q}(\Omega)}.$$

Proof. From Theorem 4.3 we have $W_{loc}^{1,(p,q)}(\Omega) \subset H_{loc}^{1,s}(\Omega)$ for every $1 < s < p$. Hence, it follows via Evans [12, p. 290, Exercise 6] that u has a version $\overline{u} \in C(\Omega)$ that is locally Hölder continuous in Ω with exponent $1-\frac{1}{s}$. Without loss of generality we can assume that u is itself continuous in $\Omega \subset \mathbf{R}$.

For both (i) and (ii) we prove first that

$$|u(x)-u(y)| \leq C(p,q)|x-y|^{1-\frac{1}{p}}||u'||_{L^{p,q}((x,y))}$$

whenever x and y are two points in Ω such that $x < y$ and $(x,y) \subset \Omega$. Here $C(p,q)$ is the constant from (19).

The function u is assumed to be continuous in Ω and the above pointwise inequality is local; thus, in order to prove it, we can assume without loss of generality for both (i) and (ii) that $\Omega \subset \mathbf{R}$ is a bounded open interval and that u is compactly supported in Ω.

From Theorem 4.3 it follows (since Ω is assumed to be bounded) that $u \in H_0^{1,s}(\Omega)$ for every $s \in (1,p)$. Fix such an $s \in (1,p)$. Let $u_k, k \geq 1$ be a sequence in $C_0^{\infty}(\Omega)$ converging to $u \in H_0^{1,s}(\Omega)$.

Since $H_0^{1,(p,q)}(\Omega) \subset H_0^{1,s}(\Omega)$, u is continuous in Ω and $n = 1 < p < \infty$, it follows immediately that u_k in fact converges to u uniformly on Ω.

The pointwise and uniform convergence of u_k to u on Ω, the fact that u_k' converges to u' in $L^{(p,q)}(\Omega)$ and the fact that (18) holds for every $k \geq 1$ and for all $x, y \in \Omega$ with $x < y$ imply immediately by passing to the limit in (18) that

$$\begin{aligned}
|u(x)-u(y)| &\leq \int_x^y |u'(t)|dt = ||u'||_{L^1((x,y))} \\
&\leq C(p,q)|x-y|^{1-\frac{1}{p}}||u'||_{L^{p,q}((x,y))} \\
&\leq C(p,q)|x-y|^{1-\frac{1}{p}}||u'||_{L^{p,q}(\Omega)}
\end{aligned}$$

for all $x, y \in \Omega$ with $x < y$. Here $C(p,q)$ is the constant from (19). This finishes the proof of the desired pointwise inequality.

Now we prove claim (i). Since $u' \in L^{(p,q)}(\Omega)$, the above pointwise inequality implies that

$$|u(x)-u(y)| \leq C(p,q)|x-y|^{1-\frac{1}{p}}||u'||_{L^{p,q}(\Omega)}$$

for all $x, y \in \Omega$ with $x < y$. In particular, u is uniformly continuous on Ω.

We claim that u admits a continuous extension to $\overline{\Omega}$. This is obvious when $\Omega = \mathbf{R}$, so we can assume without loss of generality that $\Omega \neq \mathbf{R}$.

If $\Omega = (a,b)$ is a bounded interval, then it follows that u is bounded on Ω and uniformly continuous on Ω, so in this case we can indeed extend u continuously to $\overline{\Omega} = [a,b]$. We denote the extension to $\overline{\Omega} = [a,b]$ by u as well.

If $\Omega \neq \mathbf{R}$ is an unbounded interval, then Ω is either (a,∞) or $(-\infty,a)$ for some $a \in \mathbf{R}$. In both situations, u is uniformly continuous on Ω and bounded near $x = a$, so in this case we can also extend u continuously to the unbounded set $\overline{\Omega} = \Omega \cup \{a\}$. We denote this continuous extension to $\overline{\Omega}$ by u as well.

The above pointwise inequality and the continuity of u on $\overline{\Omega}$ imply that

$$|u(x)-u(y)| \leq C(p,q)|x-y|^{1-\frac{1}{p}}||u'||_{L^{p,q}(\Omega)}$$

for all $x, y \in \overline{\Omega}$ with $x < y$. This finishes the proof of claim (i).

Now we prove claim (ii). The first part of claim (ii) follows immediately from (i).

Assume now that $u \in W^{1,(p,q)}_{loc}(\Omega)$, Ω is an interval and $u' \in L^{(p,q)}(\Omega)$. By mimicking the argument from the proof of claim (i), we see that u admits a continuous extension to $\overline{\Omega}$. If we denote that extension by u as well, we see that

$$|u(x) - u(y)| \le C(p,q)|x-y|^{1-\frac{1}{p}}||u'||_{L^{p,q}(\Omega)}$$

for all $x, y \in \overline{\Omega}$ with $x < y$. This finishes the proof of claim (ii).

Now we prove claim (iii). If $\Omega \subset \mathbf{R}$ is an interval, then claim (iii) follows obviously from (i).

Suppose now that Ω is not an interval. Let U be the smallest open interval containing Ω. That is, $U = (a, b)$, where

$$a = \inf_{x \in \Omega} x \text{ and } b = \sup_{x \in \Omega} x.$$

Let $\widetilde{u}$ be the extension of u by 0 to U. Then $\widetilde{u} \in H_0^{1,(p,q)}(U)$ via Proposition 5.2. Thus, claim (iii) holds for $\widetilde{u} \in H_0^{1,(p,q)}(U)$.

We notice that $||\widetilde{u}'||_{L^{p,q}(U)} = ||u'||_{L^{p,q}(\Omega)}$. Moreover, it is easy to see that u is in $C^{0,1-\frac{1}{p}}(\overline{\Omega})$ if and only if $\widetilde{u}$ is in $C^{0,1-\frac{1}{p}}(\overline{U})$ with

$$[\widetilde{u}]_{0,1-\frac{1}{p};\overline{U}} = [u]_{0,1-\frac{1}{p};\overline{\Omega}}.$$

Thus, claim (iii) holds also for $u \in H_0^{1,(p,q)}(\Omega)$ when Ω is not an interval. This finishes the proof of the theorem.

□

Now we prove (among other things) that if $1 < n < p < \infty$ and $1 \le q \le \infty$, then the spaces $H_0^{1,(p,q)}(\Omega)$ and $C_0(\Omega) \cap W^{1,(p,q)}(\Omega)$ embed into $C^{0,1-\frac{n}{p}}(\overline{\Omega})$. Since we work with functions in $H_0^{1,(p,q)}(\Omega)$ and in $C_0(\Omega) \cap W^{1,(p,q)}(\Omega)$, no regularity assumptions on $\partial\Omega$ are needed. This extends the Morrey embedding theorem to the Sobolev-Lorentz spaces in the Euclidean setting. We prove this theorem by relying on the well-known Poincaré inequality in the Euclidean setting and by invoking the classical Morrey embedding theorem for $1 < n < s < p < \infty$, proved by Evans in [12] and by Gilbarg-Trudinger in [15]. Theorem 5.6 (i) was also obtained via a different proof by Cianchi-Pick in [5]. (See Cianchi-Pick [5, Theorem 1.3]).

Theorem 5.6. *Suppose $1 < n < p < \infty$ and $1 \le q \le \infty$. Let $\Omega \subset \mathbf{R}^n$ be open.*

(i) *If $u \in W^{1,(p,q)}(\Omega)$ is compactly supported in Ω, then u has a version $\overline{u} \in C^{0,1-\frac{n}{p}}(\overline{\Omega})$ and*

$$[\overline{u}]_{0,1-\frac{n}{p};\overline{\Omega}} \le C(n,p,q)||\nabla u||_{L^{p,q}(\Omega;\mathbf{R}^n)}, \tag{20}$$

where $C(n,p,q) > 0$ is a constant that depends only on n, p, q.

(ii) *If $u \in W^{1,(p,q)}_{loc}(\Omega)$, then u has a version $\overline{u}$ that is locally Hölder continuous in Ω with exponent $1 - \frac{n}{p}$.*

(iii) *If $u \in W^{1,(p,q)}_{loc}(\mathbf{R}^n)$ and $|\nabla u| \in L^{(p,q)}(\mathbf{R}^n)$, then u has a version $\overline{u} \in C^{0,1-\frac{n}{p}}(\mathbf{R}^n)$ and*

$$[\overline{u}]_{0,1-\frac{n}{p};\mathbf{R}^n} \le C(n,p,q)||\nabla u||_{L^{p,q}(\mathbf{R}^n;\mathbf{R}^n)},$$

where $C(n,p,q)$ is the constant from (20).

(iv) *If $u \in H_0^{1,(p,q)}(\Omega)$, then u has a version $\overline{u} \in C^{0,1-\frac{n}{p}}(\overline{\Omega})$ and*

$$[\overline{u}]_{0,1-\frac{n}{p};\overline{\Omega}} \le C(n,p,q)||\nabla u||_{L^{p,q}(\Omega;\mathbf{R}^n)},$$

where $C(n,p,q)$ is the constant from (20).

Proof. Let $s \in (n, p)$ be fixed. We have via Gilbarg-Trudinger [15, Theorem 7.17] and via Theorem 4.3 that $W^{1,(p,q)}_{loc}(\Omega)$ embeds into the space of locally Hölder continuous functions in Ω with exponent $1 - \frac{n}{s}$. Thus, if $u \in W^{1,(p,q)}_{loc}(\Omega)$ with $1 < n < p < \infty$, we can assume without loss of generality throughout the proof of this theorem (after possibly redefining u on a subset of Ω of Lebesgue measure 0) that u is in fact locally Hölder continuous in Ω with exponent $1 - \frac{n}{s}$.

We prove now (i). Suppose that $u \in W^{1,(p,q)}(\Omega)$ is compactly supported in Ω. Then we can assume without loss of generality that Ω is bounded. Since u is compactly supported in Ω and u is locally Hölder continuous in Ω with exponent $1 - \frac{n}{s}$, it follows in fact that u can be extended continuously by 0 on $\partial\Omega$ and this extension (denoted by u as well) is in fact Hölder continuous in $\overline{\Omega}$ with exponent $1 - \frac{n}{s}$, where $1 < n < s < p$.

We extend u by 0 to $\mathbf{R}^n \setminus \Omega$ and we denote this extension by v. Since $u \in C_0(\Omega) \cap W^{1,(p,q)}(\Omega)$, it follows immediately from the definition of v that $v \in C_0(\mathbf{R}^n) \cap W^{1,(p,q)}(\mathbf{R}^n)$ and

$$\nabla v(x) = \begin{cases} \nabla u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbf{R}^n \setminus \Omega. \end{cases}$$

Moreover, since $u \in C^{0,1-\frac{n}{s}}(\overline{\Omega})$, it is easy to see that $v \in C^{0,1-\frac{n}{s}}(\mathbf{R}^n)$ with

$$[v]_{0,1-\frac{n}{s};\mathbf{R}^n} = [u]_{0,1-\frac{n}{s};\overline{\Omega}}.$$

It is also easy to see that $v \in C^{0,1-\frac{n}{p}}(\mathbf{R}^n)$ if and only if $u \in C^{0,1-\frac{n}{p}}(\overline{\Omega})$ with

$$[v]_{0,1-\frac{n}{p};\mathbf{R}^n} = [u]_{0,1-\frac{n}{p};\overline{\Omega}}.$$

It is enough to show that

$$[v]_{0,1-\frac{n}{p};\mathbf{R}^n} \leq C(n,p,q)||\nabla v||_{L^{p,q}(\mathbf{R}^n;\mathbf{R}^n)}.$$

Let $x \neq y$ be two points from $\mathbf{R}^n$ and let a be the midpoint of the segment connecting x and y. Let $R = |x - y|$.

For every integer $j \geq 0$ let $B_{x,j} = B(x, 2^{-j-1}R)$ and $B_{y,j} = B(y, 2^{-j-1}R)$. Let $B_a = B(a, R)$. It is easy to see that $B_{x,0} \cup B_{y,0} \subset B_a$.

Since v is continuous in $\mathbf{R}^n$, all the points in $\mathbf{R}^n$ are Lebesgue points for v. Thus,

$$v(x) = \lim_{j\to\infty} v_{B_{x,j}} \text{ and } v(y) = \lim_{j\to\infty} v_{B_{y,j}}.$$

Hence

$$\begin{aligned} v(x) - v(y) \quad = \quad & \left((v_{B_{x,0}} - v_{B_a}) + \sum_{j=1}^{\infty} \frac{1}{|B_{x,j+1}|} \int_{B_{x,j+1}} (v(z) - v_{B_{x,j}})\, dz \right) \\ & - \left((v_{B_{y,0}} - v_{B_a}) + \sum_{j=1}^{\infty} \frac{1}{|B_{y,j+1}|} \int_{B_{y,j+1}} (v(z) - v_{B_{y,j}})\, dz \right). \end{aligned}$$

This implies

$$\begin{aligned} |v(x) - v(y)| \quad \leq \quad & \left(|v_{B_{x,0}} - v_{B_a}| + \sum_{j=1}^{\infty} \frac{1}{|B_{x,j+1}|} \int_{B_{x,j+1}} |v(z) - v_{B_{x,j}}|\, dz \right) \\ & + \left(|v_{B_{y,0}} - v_{B_a}| + \sum_{j=1}^{\infty} \frac{1}{|B_{y,j+1}|} \int_{B_{y,j+1}} |v(z) - v_{B_{y,j}}|\, dz \right). \end{aligned}$$

Since $v \in W^{1,(p,q)}(\mathbf{R}^n)$ is compactly supported in $\mathbf{R}^n$, then via Theorem 4.3 we have $v \in H_0^{1,s}(\mathbf{R}^n)$. Thus, via Poincaré's inequality, we have

$$\frac{1}{B(w,r)} \int_{B(w,r)} |v(z) - v_{B(w,r)}| dz \le C(n) r \frac{1}{B(w,r)} \int_{B(w,r)} |\nabla v(z)| dz \tag{21}$$

for every $w \in \mathbf{R}^n$ and every $r > 0$, where $C(n) > 0$ is a constant that depends only on n.

Since $B_{x,0} \subset B_a$, we have via Hölder's inequality for Lorentz spaces (see Theorem 3.7) and Poincaré's inequality (21)

$$\begin{aligned}
|v_{B_{x,0}} - v_{B_a}| &= \frac{1}{|B_{x,0}|} \left| \int_{B_{x,0}} (v(z) - v_{B_a})\, dz \right| \\
&\le \frac{1}{|B_{x,0}|} \int_{B_{x,0}} |v(z) - v_{B_a}|\, dz \\
&\le \frac{2^n}{|B_a|} \int_{B_a} |v(z) - v_{B_a}|\, dz \\
&\le C(n) R \frac{1}{|B_a|} \int_{B_a} |\nabla v(z)|\, dz \\
&\le C(n,p,q) R \left(\frac{||\nabla v||^p_{L^{p,q}(B_a;\mathbf{R}^n)}}{|B_a|} \right)^{1/p} \\
&= C(n,p,q) R^{1-\frac{n}{p}} ||\nabla v||_{L^{p,q}(B_a;\mathbf{R}^n)} \\
&= C(n,p,q) R^{1-\frac{n}{p}} ||\nabla u||_{L^{p,q}(\Omega \cap B_a;\mathbf{R}^n)} \\
&\le C(n,p,q) R^{1-\frac{n}{p}} ||\nabla u||_{L^{p,q}(\Omega;\mathbf{R}^n)} \\
&= C(n,p,q) R^{1-\frac{n}{p}} ||\nabla v||_{L^{p,q}(\mathbf{R}^n;\mathbf{R}^n)}.
\end{aligned}$$

Similarly, since $B_{y,0} \subset B_a$, we obtain (after an almost identical reasoning, by replacing $B_{x,0}$ with $B_{y,0}$)

$$\begin{aligned}
|v_{B_{y,0}} - v_{B_a}| &\le C(n,p,q) R^{1-\frac{n}{p}} ||\nabla u||_{L^{p,q}(\Omega \cap B_a;\mathbf{R}^n)} \\
&\le C(n,p,q) R^{1-\frac{n}{p}} ||\nabla u||_{L^{p,q}(\Omega;\mathbf{R}^n)}.
\end{aligned}$$

We want to obtain upper estimates for

$$\begin{aligned}
|u_{B_{x,j+1}} - u_{B_{x,j}}| &= \frac{1}{|B_{x,j+1}|} \left| \int_{B_{x,j+1}} (u(z) - u_{B_{x,j}})\, dz \right| \text{ and} \\
|u_{B_{y,j+1}} - u_{B_{y,j}}| &= \frac{1}{|B_{y,j+1}|} \left| \int_{B_{y,j+1}} (u(z) - u_{B_{y,j}})\, dz \right|
\end{aligned}$$

for all $j \ge 0$.

For all $j \ge 0$ we only do the estimate for $|u_{B_{x,j+1}} - u_{B_{x,j}}|$ because we would use an almost identical reasoning to obtain the estimate for $|u_{B_{x,j+1}} - u_{B_{x,j}}|$.

Since $B_{x,j+1} \subset B_{x,j} \subset B_a$ for all $j \geq 0$, we have via Hölder's inequality for Lorentz spaces (see Theorem 3.7) and Poincaré's inequality (21)

$$\begin{aligned}
|v_{B_{x,j+1}} - v_{B_{x,j}}| &= \frac{1}{|B_{x,j+1}|}\left|\int_{B_{x,j+1}} (v(z) - v_{B_{x,j}})\, dz\right| \\
&\leq \frac{1}{|B_{x,j+1}|}\int_{B_{x,j+1}} |v(z) - v_{B_{x,j}}|\, dz \\
&\leq \frac{2^n}{|B_{x,j}|}\int_{B_{x,j}} |v(z) - v_{B_{x,j}}|\, dz \\
&\leq C(n)R\frac{1}{|B_{x,j}|}\int_{B_{x,j}} |\nabla v(z)|\, dz \\
&\leq C(n,p,q)2^{-j}R\left(\frac{||\nabla v||^p_{L^{p,q}(B_{x,j};\mathbf{R}^n)}}{|B_{x,j}|}\right)^{1/p} \\
&\leq C(n,p,q)2^{-j}R\left(\frac{||\nabla v||^p_{L^{p,q}(B_a;\mathbf{R}^n)}}{|B_{x,j}|}\right)^{1/p} \\
&= C(n,p,q)(2^{-j}R)^{1-\frac{n}{p}}||\nabla v||_{L^{p,q}(B_a;\mathbf{R}^n)} \\
&\leq C(n,p,q)(2^{-j}R)^{1-\frac{n}{p}}||\nabla v||_{L^{p,q}(\mathbf{R}^n;\mathbf{R}^n)}.
\end{aligned}$$

By summing the above inequalities and taking into account that $|x-y| = R$, we have

$$\begin{aligned}
|v(x) - v(y)| &\leq C(n,p,q)||\nabla v||_{L^{p,q}(B_a;\mathbf{R}^n)}\left(\sum_{j=0}^{\infty}(2^{-j}R)^{1-\frac{n}{p}}\right) \\
&= C(n,p,q)R^{1-\frac{n}{p}}||\nabla v||_{L^{p,q}(B_a;\mathbf{R}^n)} \\
&= C(n,p,q)|x-y|^{1-\frac{n}{p}}||\nabla v||_{L^{p,q}(B_a;\mathbf{R}^n)} \\
&\leq C(n,p,q)|x-y|^{1-\frac{n}{p}}||\nabla v||_{L^{p,q}(\mathbf{R}^n;\mathbf{R}^n)}.
\end{aligned}$$

Claim (i) holds with constant $C(n,p,q)$ from the last line in the above sequence of inequalities. This finishes the proof of claim (i).

We prove now claim (ii). Let $\Omega' \subset\subset \Omega$ be an open subset of Ω and let $u \in W^{1,(p,q)}_{loc}(\Omega)$. We choose a cut-off function $\varphi \in C_0^\infty(\Omega)$ such that $0 \leq \varphi \leq 1$ and such that $\varphi = 1$ in $\overline{\Omega'}$. Then $u\varphi$ is compactly supported in Ω and via Lemma 4.9 we have $u\varphi \in W^{1,(p,q)}(\Omega)$. From part (i) we have that $u\varphi$ is Hölder continuous in $\overline{\Omega}$ with exponent $1 - \frac{n}{p}$ and

$$[u\varphi]_{0,1-\frac{n}{p};\overline{\Omega}} \leq C(n,p,q)||\nabla(u\varphi)||_{L^{p,q}(\Omega;\mathbf{R}^n)},$$

where $C(n,p,q) > 0$ is the constant from (20). Since $\Omega' \subset\subset \Omega$ and $\varphi = 1$ in $\overline{\Omega'}$, it follows that $u\varphi = u$ in $\overline{\Omega'}$. Thus, u is Hölder continuous in $\overline{\Omega'}$ with exponent $1 - \frac{n}{p}$ and

$$\begin{aligned}
[u]_{0,1-\frac{n}{p};\overline{\Omega'}} &= [u\varphi]_{0,1-\frac{n}{p};\overline{\Omega'}} \leq [u\varphi]_{0,1-\frac{n}{p};\overline{\Omega}} \\
&\leq C(n,p,q)||\nabla(u\varphi)||_{L^{p,q}(\Omega;\mathbf{R}^n)},
\end{aligned}$$

where $C(n,p,q) > 0$ is the constant from (20). This finishes the proof of (ii).

We prove now claim (iii). We use the notation from part (i).

Let $x \neq y$ be two points in $\mathbf{R}^n$, let a be the midpoint of the segment $[x,y]$ and let $R = |x-y|$. Let $\varphi_{x,y} \in C_0^\infty(\mathbf{R}^n)$ be a function such that $0 \leq \varphi_{x,y} \leq 1$ and such that $\varphi_{x,y} = 1$ on $B_a := B(a,R)$.

By running the argument from (i) with the function $u\varphi_{x,y} \in W^{1,(p,q)}(\Omega)$ that is compactly supported in $\mathbf{R}^n$ we obtain, since $\varphi_{x,y} = 1$ on $B_a \ni \{x, y\}$

$$\begin{aligned} |u(x) - u(y)| &= |(u\varphi_{x,y})(x) - (u\varphi_{x,y})(y)| \\ &\le C(n,p,q)||\nabla(u\varphi_{x,y})||_{L^{p,q}(B_a;\mathbf{R}^n)} \\ &= C(n,p,q)||\nabla u||_{L^{p,q}(B_a;\mathbf{R}^n)} \le C(n,p,q)||\nabla u||_{L^{p,q}(\mathbf{R}^n;\mathbf{R}^n)}, \end{aligned}$$

where $C(n,p,q)$ is the constant from (20) and from the last line of the last sequence of inequalities in the proof of claim (i). This finishes the proof of claim (iii).

We prove now claim (iv). We have to consider the cases $\Omega = \mathbf{R}^n$ and $\Omega \subsetneq \mathbf{R}^n$ separately.

Suppose first that $\Omega = \mathbf{R}^n$. In this case the claim follows immediately from (iii) because the membership of u in $H_0^{1,(p,q)}(\mathbf{R}^n)$ implies that u is in $W_{loc}^{1,(p,q)}(\mathbf{R}^n)$ and $|\nabla u| \in L^{(p,q)}(\mathbf{R}^n)$.

Suppose now that $\Omega \subsetneq \mathbf{R}^n$. Let v be the extension by 0 of u to $\mathbf{R}^n \setminus \Omega$. We claim that v is continuous in $\mathbf{R}^n$.

Indeed, let $(u_k)_{k\ge1} \subset C_0^\infty(\Omega)$ be a sequence of functions such that u_k converges to u in $H_0^{1,(p,q)}(\Omega)$ and pointwise almost everywhere in Ω. For every $k \ge 1$ let v_k be the extension by 0 of u_k to $\mathbf{R}^n \setminus \Omega$. We see immediately that $(v_k)_{k\ge1} \subset C_0^\infty(\mathbf{R}^n)$ and that v_k converges to v pointwise almost everywhere in $\mathbf{R}^n$. Moreover, from Proposition 5.2, it follows that v_k converges to v in $H_0^{1,(p,q)}(\mathbf{R}^n)$.

Since the sequence $(u_k)_{k\ge1} \subset C_0^\infty(\Omega)$ converges to u in $H_0^{1,(p,q)}(\Omega)$, u is continuous in Ω, and $1 < n < p < \infty$, it follows immediately that in fact u_k converges to u uniformly on compact subsets of Ω. In particular, u_k converges pointwise to u everywhere in Ω.

From this, the definition of v and of the functions v_k and from the fact that $v_k = v = 0$ everywhere on $\mathbf{R}^n \setminus \Omega$ for all $k \ge 1$, it follows that the sequence v_k converges pointwise to v everywhere in $\mathbf{R}^n$. Since the sequence $(v_k)_{k\ge1} \subset C_0^\infty(\mathbf{R}^n)$ converges to v in $H_0^{1,(p,q)}(\mathbf{R}^n)$ and pointwise in $\mathbf{R}^n$ and since $1 < n < p < \infty$, it follows immediately that in fact $v \in C(\mathbf{R}^n)$ and the sequence v_k converges to v uniformly on compact subsets of $\mathbf{R}^n$. Thus, we proved that v is continuous in $\mathbf{R}^n$. If we denote the extension of u by 0 to $\partial\Omega$ by u as well, the above argument proved that $u \in C(\overline{\Omega})$.

We see now that the claim (iv) holds for v via (iii). Since $v \in C^{0,1-\frac{n}{p}}(\mathbf{R}^n)$, and v is the continuous extension by 0 of the function $u \in C(\overline{\Omega})$ to $\mathbf{R}^n$, it follows immediately that $u \in C^{0,1-\frac{n}{p}}(\overline{\Omega})$ and

$$[u]_{0,1-\frac{n}{p};\overline{\Omega}} = [v]_{0,1-\frac{n}{p};\mathbf{R}^n}.$$

This implies immediately that the claim (iv) holds for u as well, because

$$||\nabla u||_{L^{p,q}(\Omega;\mathbf{R}^n)} = ||\nabla v||_{L^{p,q}(\mathbf{R}^n;\mathbf{R}^n)}.$$

This finishes the proof of the theorem.

□

Theorems 5.5 and 5.6 together with Proposition 4.7 yield the following corollary.

Corollary 5.7. *Suppose $1 \le n < p < \infty$, where n is an integer. Let $u_p : \mathbf{R}^n \to \mathbf{R}$, $u_p(x) = |x|^{1-\frac{n}{p}}$. Then u_p is Hölder continuous in $\mathbf{R}^n$ with exponent $1 - \frac{n}{p}$.*

Proof. We proved in Proposition 4.7 that $u_p \in W_{loc}^{1,(p,\infty)}(\mathbf{R}^n)$ for all $p \in (1,\infty)$. The claim follows immediately by invoking Theorem 5.5 (ii) for $n = 1$ and respectively Theorem 5.6 (iii) for $n > 1$. One can also see that the claim holds via a direct and easy computation. □

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References

[1] C. Bennett and R. Sharpley. *Interpolation of Operators*. Academic Press, 1988.

[2] J. Björn. Boundary continuity for quasiminimizers on metric spaces. *Illinois J. Math.*, **46** (2002), no. 2, 383–403.

[3] J. Cheeger. Differentiability of Lipschitz functions on metric spaces. *Geom. Funct. Anal.*, **9** (1999), 428–517.

[4] A. Cianchi, L. Pick. Sobolev embeddings into BMO, VMO, and L_∞. *Ark. Mat.*, **36** (1998), 317–340.

[5] A. Cianchi, L. Pick. Sobolev embeddings into spaces of Campanato, Morrey, and Hölder type. *J. Math. Anal. Appl.*, **282** (2003), 128–150.

[6] Ş. Costea. *Strong A_∞-weights and scaling invariant Besov and Sobolev-Lorentz capacities*, Ph.D thesis, University of Michigan, 2006.

[7] Ş. Costea. Scaling invariant Sobolev-Lorentz capacity on $\mathbb{R}^n$. *Indiana Univ. Math. J.*, **56** (2007), no. 6, 2641–2669.

[8] Ş. Costea. Sobolev capacity and Hausdorff measures in metric measure spaces, *Ann. Acad. Sci. Fenn. Math*, **34** (2009), no. 1, 179-194.

[9] Ş. Costea *Scaling Besov and Sobolev-Lorentz capacities in the Euclidean setting*, Publishing House of the Romanian Academy, Bucharest, 2012, xii+114 pages.

[10] Ş. Costea and V. Maz'ya. Conductor inequalities and criteria for Sobolev-Lorentz two-weight inequalities, *Sobolev Spaces in Mathematics II. Applications in Analysis and Partial Differential Equations*, p. 103-121, Springer, (2009).

[11] Ş. Costea and M. Miranda Jr. (2012). Newtonian Lorentz metric spaces. *Illinois J. Math.* **56** no. 2, 579–616.

[12] L.C. Evans. *Partial Differential Equations.* American Mathematical Society, 1998.

[13] H. Federer. *Geometric Measure Theory.* Springer-Verlag, 1969.

[14] B. Franchi, P. Hajłasz and P. Koskela. Definitions of Sobolev classes on metric spaces. *Ann. Inst. Fourier (Grenoble)*, **49** (1999), 1903–1924.

[15] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, 2nd edition, 1983.

[16] P. Hajłasz. Sobolev spaces on an arbitrary metric space. *Potential Anal.*, **5** (1996), 403–415.

[17] P. Hajłasz. Sobolev spaces on metric-measure spaces. (Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)) *Contemp. Math.*, **338** (2003) Amer. Math. Soc., Providence, RI, 173–218.

[18] J. Heinonen. *Lectures on Analysis on Metric Spaces.* Springer-Verlag, 2001.

[19] J. Heinonen, T. Kilpeläinen and O. Martio. *Nonlinear Potential Theory of Degenerate Elliptic Equations.* Oxford University Press, 1993.

[20] J. Heinonen and P. Koskela. Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.*, **181** (1998), 1–61.

[21] R. Hunt. On $L(p,q)$ measure spaces. *Enseignement Math.*, **12** (1966), no. 2, 249–276

[22] J. Kauhanen, P. Koskela and J. Malý. On functions with derivatives in a Lorentz space. *Manuscripta Math.*, **100** (1999), no. 1, 87–101.

[23] J. Kinnunen and O. Martio. The Sobolev capacity on metric spaces. *Ann. Acad. Sci. Fenn. Math*, **21** (1996), no. 2, 367–382.

[24] J. Kinnunen and O. Martio. Choquet property for the Sobolev capacity in metric spaces. In *Proceedings of the conference on Analysis and Geometry held in Novosibirsk*, pages 285–290, 2000.

[25] J. Malý, D. Swanson, and W. P. Ziemer. Fine behavior of functions whose gradients are in an Orlicz space. *Studia Math.* **190** (2009), no. 1, 33–71.

[26] V. Maz'ya. *Sobolev Spaces.* Springer-Verlag, 1985.

[27] N. Shanmugalingam. Newtonian spaces: An extension of Sobolev spaces to metric measure spaces. *Rev. Mat. Iberoamericana*, **16** (2000), no. 2, 243–279.

[28] N. Shanmugalingam. Harmonic functions on metric spaces. *Illinois J. Math.*, **45** (2001), no. 3, 1021–1050.

[29] E. Stein, G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton University Press, 1975.

[30] W.P. Ziemer. *Weakly Differentiable Functions.* Springer-Verlag, 1989.

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